Poincaré models for the hyperbolic plane

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1 Abstract

In the following we will investigate two representations of the hyperbolic plane $\mathcal{H}^2$: the disc model and the Upper Half-plane model. In particular we will look at the geodesics in the two representations, and especially discuss the distance between two arbitrary points in the disc model. Last but not least it will be shown that the Gaussian curvature of the hyperbolic plane is constantly -1, by computing the Gaussian curvature in the Upper Half-plane model.

2 The Upper Half-plane

Let

$$\mathcal{H}^2 = \{z = x + iy \in \mathbb{C} : y > 0\}$$

be given the metric

$$(g_{ij}(z)) = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix},$$

a slightly manipulated version of the standard Euclidean metric. The goal of this chapter is to find the geodesics of the Upper Half-Plane. To do that we'll show that the group

$$\mathrm{Gl}(2, \mathbb{R})^+ = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc > 0 \right\}$$

acts on $\mathcal{H}^2$ as a group of isometries and it also acts transitively on $\mathcal{H}^2$, because then we have that $\mathcal{H}^2$ is geodesically complete by a proposition proved at the end of this note.

The main result of this chapter is to show that the geodesics in $\mathcal{H}^2$ are the half circles and half lines perpendicular to the line $y = 0$. In particular $\mathcal{H}^2$ is geodesically complete, i.e. all geodesics are defined on all of $\mathbb{R}$.

First a proof of the statement about $\mathrm{Gl}(2, \mathbb{R})^+$ acting as isometries on $\mathcal{H}^2$. We want a useable expression for the metric given two tangent vectors to $\mathcal{H}^2$. We identify the tangent vectors in $\mathcal{H}^2$ by complex numbers, so for a point $z \in \mathcal{H}^2$ the inner product induced by the metric in the tangent space $T_z\mathcal{H}^2$ is given by

$$\langle v, w \rangle_z = \frac{1}{(\mathrm{Im}\, z)^2} \mathrm{Re}(v\bar{w}), \quad v, w \in \mathbb{C}.$$

We have to specify how $\mathrm{Gl}(2, \mathbb{R})^+$ acts on $\mathcal{H}^2$. So for a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ consider the map $h : \mathcal{H}^2 \to \mathcal{H}^2$ given by $h(z) = \frac{az+b}{cz+d}$. First we need to see that h actually goes into $\mathcal{H}^2$, so the imaginary part of $h(z)$ has to be positive.

$$\begin{aligned} \mathrm{Im}(h(z)) &= \frac{1}{2i} \left(\frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d} \right) = \frac{1}{2i} \frac{(az+b)(c\bar{z}+d) - (a\bar{z}+b)(cz+d)}{|cz+d|^2} \\ &= \frac{\Delta\, \mathrm{Im}\, z}{|cz+d|^2} > 0 \quad \text{for } z \in \mathcal{H}^2 \text{ and } \Delta = ad - bc > 0. \end{aligned}$$

For h to be an isometry it has to be a diffeomorphism. It and its inverse are holomorphic maps so it is clearly a diffeomorphism, and then the pullback of the metric g under h should be equal to g, i.e. $h^*g = g$.

Generally for a holomorphic map $f(z) = u(x+iy)+iv(x+iy)$ we have the *Cauchy-Riemann* equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} =: \lambda$ and $-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} =: \mu$, and the pushforward of f on a vector is given by

$$\begin{pmatrix} u \\ v \end{pmatrix}_* \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \lambda a - \mu b \\ \mu a + \lambda b \end{pmatrix},$$

but $(\lambda + i\mu)(a + ib) = \lambda a - \mu b + i(\mu a + \lambda b)$, so the differential at a point z is given by multiplication by the complex derivative of h.

$$h'(z) = \frac{(cz + d)a - (az + b)c}{(cz + d)^2} = \frac{\Delta}{(cz + d)^2},$$

where $\Delta = ad - bc$ is the determinant.

Therefore let $v, w \in \mathbb{C}$, then

$$h^* g_z(v, w) = \langle h_* v, h_* w\rangle_{h(z)} = \frac{1}{(\operatorname{Im} h(z))^2} \operatorname{Re}\left(g'(z)v\overline{g'(z)}\bar{w}\right)$$
$$= \frac{|g'(z)|^2}{(\operatorname{Im} g(z))^2} \operatorname{Re}(v\bar{w}) = \frac{1}{(\operatorname{Im} z)^2} \operatorname{Re}(v\bar{w}) = \langle v, w\rangle_z = g_z(v, w),$$

and h is an isometry. So with the matrix beeing arbitrary, $\mathrm{Gl}(2, \mathbb{R})^+$ acts as a group of isometries on $\mathcal{H}^2$.

Now we are able to find the geodesics, and the first curve we want to show beeing a geodesic is the half y-axis,

$$l = \{iy : y > 0\}.$$

To see this we use the reflection $r : \mathcal{H}^2 \to \mathcal{H}^2$ given by $r(x + iy) = -x + iy$, which is easily seen to be an isometry, though not covered by the actions of $\mathrm{Gl}(2, \mathbb{R})^+$, which has l as the fixed points. If we choose the start vector $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ it will also be invariant under r, as the differential of a linear map is the linear map itself, and in $\mathbb{R}^2$ r is just multiplication with $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Since r is an isometry, then if $l(t)$ is a geodesic then $r(l(t))$ is also a geodesic, and if we choose the initial data $i \in \mathbb{C}$ as start point and v as a start vector, then if $l(t)$ is parametrized by arc-length then, $r(l(t)) = l(t)$, and the geodesic will stay in $l(t)$ to all times, that it might be defined. By the way, this is a very standard way of finding geodesics.

As noted our geodesic has to be parametrized by arc length, this gives us the ability to give an explicit expression for the geodesic. So $|l'(t)|_{\mathcal{H}^2} = 1$ for all t. $l(t) = iy(t)$ and $l'(t) = iy'(t)$ so

$$|l'(t)|_{\mathcal{H}^2} = \frac{|y'(t)|}{\operatorname{Im} l(t)} = \frac{|y'(t)|}{y(t)},$$

so $y(t) = |y'(t)|$ which means that $y(t) = \pm y'(t)$. The solution to the differential equation with positive sign points up and the one with negative sign points down. We just look at the equation with positive sign. It gives the solution $l(t) = C + e^t$, but $l(0) = i$, so $C = 0$, and $l(t) = ie^t$, is a geodesic in $\mathcal{H}^2$ defined for all $t \in \mathbb{R}$.

In order to get geodesics in all directions through the point i, we need to find isometries that have i as fixed point, so let $h \in \mathrm{Gl}(2, \mathbb{R})^+$ be the isometry given by the rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \theta \in \mathbb{R}.$$

Then i is a fix point

$$h(i) = \frac{i\cos\theta - \sin\theta}{\sin\theta + \cos\theta} = i,$$

and the differential of $h(z)$ is given by multiplication by

$$h'(i) = \frac{1}{(i \sin\theta + \cos\theta)^2} = e^{-2i\theta},$$

so $h(l)$ is a geodesic through i pointing in the direction determined by the angle $\frac{\pi}{2} - 2\theta$. In order to get geodesics trough any other point just notice that $\mathrm{Gl}(2,\mathbb{R})^+$ acts transitively on $\mathcal{H}^2$. In fact if $z = bi + a$, $b > 0$ then $z = h(i)$ for h given by the matrix

$$\begin{pmatrix} b & a \\ 0 & 1 \end{pmatrix}, \quad \Delta = b > 0.$$

In all cases the geodesics are the images of l under som Möbius transformation as in the action of $\mathrm{Gl}(2,\mathbb{R})^+$. The Möbius transformation take circles (or lines) to circles (or liness) and also preserve angles. Therefore, and since the real axis is mapped onto itself, the images will always be either a circle perpendicular to the real axis or a line perpendicular to the real axis. You could think, that the geodesic only needs to be perpendicular in one end to the real axis, but by rotating the geodesic π radians we see that also the other end must be perpendicular to the real axis.

Remark 2.1. The mother of all the geodesics in $\mathcal{H}^2$ was $\gamma(t) = e^{it}$, which is defined for all $t \in \mathbb{R}$. By the transformations we use to get the rest of the geodesics we do not alter with the timedependence. Therefore the upper half-plane is geodesically complete, and by Hopf-Rinows theorem also complete as a metric space.

Now we have established all the facts about geodesics in the upper half-plane. Now we want to transfer these results to *the disc model.*

3 The disc model

Let $\mathcal{D} \subset \mathbb{C}$ be the unit disc $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the Riemannian metric, g, given by

$$g(v, w) = \langle v, w \rangle_z = \frac{4 \operatorname{Re}(v\bar{w})}{1 - |z|^2}^2.$$

$\mathcal{D}$ equipped with this Riemannian metric is the *Poincaré model for the hyperbolic plane.* We will tie this description of the hyperbolic plane closely together with the upper half plane, $\mathcal{H}^2$.

If we can find an isometry between $\mathcal{D}$ and $\mathcal{H}^2$, we can transfer our results about geodesics in $\mathcal{H}^2$ to $\mathcal{D}$. We will look at the *Cayley-transform*, $c : \mathcal{D} \to \mathcal{H}$ and see that it is an isometry.

The cayley-transform is given by $c(z) = -i\frac{z+i}{z-i}$, and to see that c actually goes into $\mathcal{H}^2$ we just have to look at the imaginary part of $c(z)$:

$$\operatorname{Im}(c(z)) = \frac{1}{2i}\left(-i\frac{z+i}{z-i} - i\frac{\bar{z}-i}{\bar{z}+i}\right) = -\frac{1}{2}\left(\frac{(z+i)(\bar{z}+i) + (\bar{z}-i)(z-i)}{|z-i|^2}\right)$$
$$= \frac{1-|z|^2}{|z-i|^2} > 0 \quad \text{for } z \in \mathcal{D}.$$

We also need to verify that c is a diffeomorphism, that is both c and c^{-1} are holomorphic. By definition the cayley-transform is clearly holomorphic. The inverse cayley-transform can easily be calculated to be

$$c^{-1}(w) = \frac{w-i}{1-iw},$$

which also clearly is a holomorphic map, so the cayley-transform is a diffeomorphism.

To show that c is an isometry, the pullback of the metric $\tilde{g}$ to $\mathcal{D}$ should be equal to g: $c^*\tilde{g} = g$. To make this calculation we need the pushforward of c, and because we only have one complex variable we should just calculate the derivative of c with respect to z, and then $c_* w = c'(z)w$:

$$c'(z) = \frac{-2}{(z-i)^2}.$$

Now let us show that the cayley-transform is an isometry. Let $v, w \in \mathbb{C}$

$$\begin{aligned} c^*\tilde{g}(v,w) = \langle c_* v, c_* w\rangle_{c(z)} &= \frac{|z-i|^4}{(1-|z|^2)^2} \operatorname{Re}\left(\frac{4v\bar{w}}{|z-i|^4}\right) \\ &= \frac{4}{(1-|z|^2)^2}\operatorname{Re}(v\bar{w}) = \langle v, w\rangle_z = g(v,w), \end{aligned}$$

so $c^*\tilde{g} = g$ and c is an isometry.

So now we have a method to transfer geodesics from $\mathcal{H}^2$ to $\mathcal{D}$, so that they remain geodesics in $\mathcal{D}$. The first kind of geodesics we look at are the ones through $i \in \mathcal{H}^2$, because they will be geodesics through $c^{-1}(i) = 0 \in \mathcal{D}$. In Chapter 2 the fundamental geodesic was $\gamma(t) = ie^t$, and because c^{-1} is an isometry $\psi(t) = c^{-1} \circ \gamma(t)$ is a geodesic in $\mathcal{D}$ through 0, with the equation

$$\psi(t) = \frac{ie^t - i}{1+e^t} = i\frac{e^t-1}{e^t+1} = i\frac{e^{t/2}-e^{-t/2}}{e^{t/2}+e^{-t/2}} = i\tanh(t/2).$$

Taking ψ to the limit $\psi(t) \to i$ for $t \to \infty$ and $\psi(t) \to -i$ for $t \to -\infty$, we see that ψ is represented in $\mathcal{D}$ as a vertical straight line through 0.

All geodesics through i in $\mathcal{H}^2$ can be specified by rotation of the initial vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so all geodesics through $0 \in \mathcal{D}$ are given by rotations of this vertical line generated by ψ, so all geodesics through 0 in $\mathcal{D}$ are straight lines.

With γ parametrized by arc length, we get furthermore that ψ also is. This implies that if $z' \in \mathcal{D}$ is given, we can rotate it around 0 to the imaginary axis to z. Now the geodesic between 0 and z is ψ, and because ψ is parametrized by arc length $d(o,z) = t$ if $z = \psi(t)$. That is $z = \psi(t) = i\frac{e^t-1}{e^t+1}$ and with z beeing on the imaginary axis $|z| = \frac{e^t-1}{e^t+1}$, and by isolating e^t we get $e^t = \frac{|z|+1}{1-|z|}$ which implies that

$$d(o,z) = t = \log\left(\frac{|z|+1}{1-|z|}\right).$$

Now we found all the geodesics through 0 in $\mathcal{D}$ and that is a good start. But given two arbitrary points in $\mathcal{D}$, $z_{,}z_2$, we want to find the minimal geodesic between them, to give a concrete formula for the distance between the two points. Just like we did with 0 and the arbitrary point z' above. The idea is to use the transformations between $\mathcal{D}$ and $\mathcal{H}^2$ to find the geodesics. First we use the cayley-transformation to get two points $w_1 = c(z_1)$ and $w_2 = c(z_2)$ in $\mathcal{H}^2$. Now we know, that the geodesic between w_1 and w_2 are either a straight line perpendicular to the real axis or half of the arc of a circle which is perpendicular to the real axis. Let us call the geodesics intersection with the real axis for v_1 and v_2. If the geodesic is a straight line, just let v_2 be infinity. Now we use the inverse cayley-transformation to transport the geodesic and the four points back to $\mathcal{D}$. The inverse cayley-transformation is an isometry, and therefore it preserves angles, so the geodesic is perpendicular to the boundary of $\mathcal{D}$. Futhermore because c^{-1} is a Möbiustransformation it takes circles (or straigth lines) to circles (or straight lines), so the geodesic between z_1 and z_2 is either a straight line or a part of a circle, in both cases the curve is perpendicular to $\partial\mathcal{D}$. The intersection of the geodesic with $\partial\mathcal{D}$ are $b_1 = c(v_1)$ and $b_2 = c(v_2)$.

The next thing is to determine the distance between the two points. For that, we use the points corresponding points w_1, w_2, v_1 and v_2 in $\mathcal{H}^2$. Because we know, that here we can use a Möbiustransformation to move the three points w_1, v_1, v_2 to respectively $i, 0, \infty$. Now we transform back to $\mathcal{D}$ with the inverse cayley-transformation, so all in all z_1 goes to 0, b_1 goes to i, b_2 goes to $-i$ and z_2 goes to an unspecified point $z \in \mathcal{D}$. But z is not completly arbitrary, because it lies on the geodesic between 0, i and $-i$ which is the imaginary axis, so z is of the form $z = i\,|z|$. The reason we do this is, that now we know the distance between 0 and z from our previous calculations. And with the cayley- and inverse cayley-transformations beeing Möbiustransformations our total transformation of (z_1, z_2, b_2, b_1) to $(0, z, -i, i)$ is a Möbiustransformation. We know want to use the fact that the cross-ratio is invariant under Möbuistransformations to get an expression of the distance between z_1 and z_2.

It will be proved later that the cross-ratio is invariant under Möbiustransformations, but let us just assume that for the moment. Let us also assume that the Möbiustransformations we use is an isometry (that will also be proved later). The cross-ratio of four points is defined as

$$[z_1, z_2; z_3, z_4] = \frac{z_1 - z_3}{z_1 - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3}.$$

So the cross-ratio of the tuple $(0, z; -i, i)$ is

$$[0, z; -i, i] = \frac{0 - i}{0 + i}\frac{z + i}{z - i} = \frac{|z| + 1}{1 - |z|}.$$

But now because the transformation was an isometry, and the cross-ratio is invariant under the transformation we get the following formula for the distance between z_1 and z_2

$$d(z_1, z_2) = d(z, 0) = \log\left(\frac{|z| + 1}{1 - |z|}\right) = \log\left(\frac{z_1 - b_2}{z_1 - b_1} \cdot \frac{z_2 - b_1}{z_2 - b_2}\right).$$

What we need to show is, that the cross-ratio is invariant under Möbiustransformations and that the Möbiustransformations we need are isometries.

Remark 3.1. The discmodel is also complete, since the time dependence is not changed by the Cayley-transformation that takes geodesics from the upper half plane to the disc. So geodesics in $\mathcal{D}$ is defined to all times.

3.1 Möbiustransformations on $\mathcal{D}$ are isometries

The Möbiustransformations on $\mathcal{D}$ were constructed by conjugation of a Möbiustransformation on $\mathcal{H}^2$, or rather on the upper half plane as a subset of $\mathbb{R}^2$. These transformations are

$$z \mapsto \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{R} \quad ad - bc > 0,$$

and we denote them by f. We use the Cayley-transform to go back and forth between $\mathcal{D}$ and $\mathcal{H}^2$. So, if we define g as $g(z) = c^{-1} \circ f \circ c(z)$, we get a transformation from $\mathcal{D}$ to itself. The next thing is to calculate an expression for g that is easy to work with.

Let $z \in \mathcal{D}$.

$$\begin{aligned} c^{-1} \circ f \circ c(z) &= c^{-1} \circ f\left(-i\frac{z+i}{z-i}\right) = c^{-1}\left(\frac{a\left(-i\frac{z+i}{z-i}\right)+b}{c\left(-i\frac{z+i}{z-i}\right)+d}\right) \\ &= c^{-1}\left(\frac{-iaz+a+bz-bi}{-icz+c+dz-di}\right) \\ &= \frac{\frac{-iaz+a+bz-bi}{-icz+c+dz-di}-i}{1-i\left(\frac{-iaz+a+bz-bi}{-icz+c+dz-di}\right)} \\ &= \frac{z(b-c+i(-a-d))+a-d+i(-b-c)}{-z(a-d-i(-b-c))-(b-c-i(-a-d))} \\ &= -\frac{zn+m}{z\bar{m}+\bar{n}}, \end{aligned}$$

where $m = a-d+i(-b-c)$ and $n = b-c+i(-a-d)$. We also had the condition that $ad-bc>0$, and this means that $|n|^2 - |m|^2 = 4(ad-bc) > 0$.

We already know that the Cayley-transform is an isometry, so if we can just show that Möbiustransformations on $\mathcal{H}^2$ are isometries, then g is also an isometry, and we are done. But in Chapter 2 we showed that $\mathrm{Gl}(2,\mathbb{R})^+$ acts as isoemtries on $\mathcal{H}^2$, and the way this group acts on $\mathcal{H}^2$ is by Möbiustransforming the given element of $\mathcal{H}^2$ that it is acting on. In other word we showed that the Möbiustransformation in $\mathcal{D}$ is an isometry.

3.2 Cross-ratio is preserved by Möbiustransformations

We want to show that cross-ratio is preserved by general Möbiustransformations with $a,b,c,d \in \mathbb{C}$ and not in $\mathbb{R}$ as we have required so far, and just $ad-bc \neq 0$. This will especially give us that the Möbiustransformations we need preserves cross-ratio.

The cross-ratio between four points in $\mathbb{C}$ is

$$[z_1, z_2; z_3, z_4] = \frac{(z_1-z_3)}{(z_1-z_4)}\frac{(z_2-z_4)}{(z_2-z_3)}.$$

So when $f(z) = \frac{az+b}{cz+d}$ we have that

$$[f(z_1), f(z_2); f(z_3), f(z_4)] = \frac{\left(\frac{az_1+b}{cz_1+d}-\frac{az_3+b}{cz_3+d}\right)\left(\frac{az_2+b}{cz_2+d}-\frac{az_4+b}{cz_4+d}\right)}{\left(\frac{az_1+b}{cz_1+d}-\frac{az_4+b}{cz_4+d}\right)\left(\frac{az_2+b}{cz_2+d}-\frac{az_3+b}{cz_3+d}\right)}$$

First of all it is clear that you must be able to reduce this horrible expression, but secondly it is also clear that it would be a mess to write all the details. So we calculate all of them in one go. Let $i \neq j$.

$$\begin{aligned} \frac{az_i+b}{cz_i+d} - \frac{az_j+b}{cz_j+d} &= \frac{acz_iz_j + cbz_j + adz_i + bd - acz_iz_j - bd - adz_j - bcz_i}{(cz_j+d)(cz_j+d)} \\ &= \frac{z_i(ad-bc) - z_j(ad-bc)}{(cz_i+d)(cz_j+d)} \\ &= \frac{(z_i-z_j)(ad-bc)}{(cz_i+d)(cz_j+d)}. \end{aligned}$$

All of the four brackets gives this fraction – with the correct i and j's – but the only thing that do not cancel is the term $z_i - z_j$, so we see that

$$\begin{aligned}[f(z_1), f(z_2); f(z_3), f(z_4)] &= \frac{(z_1 - z_3)}{(z_1 - z_4)} \frac{(z_2 - z_4)}{(z_2 - z_3)} \\ &= [z_1, z_2; z_3, z_4],\end{aligned}$$

and the cross-ratio is invariant under Möbiustransformations.

3.3 Geodesically completeness

The following proposition is very handy and an easy way to show that a Riemannian manifold is geodesically complete, and in view of Hopf-Rinows theorem, it is then easy to show that a Riemannian manifold is metrically complete.

Let (M, g, ∇) be a Riemannian manifold with a connection ∇ – it could be the Rieammian connection, but it doesn't have to. Then M is geodesically complete if and only if every maximal geodesics is defined on all of $\mathbb{R}$. The maximal geodesic with speed V is denoted γ_V.

But before we state the proposition we need the notion of a exponential map defined on the tangent bundle to a manifold.

Define a subset Υ of TM (the tangent bundle), the domain of the exponential map, by

$$\Upsilon := \{V \in TM : \gamma_V \text{ is defined on an interval containing } [0, 1]\},$$

and then define the exponential map $\exp : \Upsilon \to M$ by

$$\exp(V) = \gamma_V(1).$$

For each $p \in M$ the restriced exponential map $\exp_p$ is the restriction of exp to the set $\Upsilon_p := \Upsilon \cap T_pM$.

The following are som properties about the exponential map that will not be proved. But proofs can be found in [Lee(1997)]

Proposition 3.2 (Properties of the Exponential Map)**.**

(a) *Υ is an open subset of TM containing the zero section, and each set Υ_p is star-shaped with respect to 0.*

(b) *For each $V \in TM$, the geodesic γ_V is given by*

$$\gamma_V(t) = \exp(tV)$$

for all t such that either side is defined

(c) *The exponential map i smooth*

(d) *Suppose M and N are Riemannian manifolds with Riemannian connections and suppose that $\varphi : M \to N$ is an isomorphism, then $\varphi_*(\Upsilon_p(M)) \subset \Upsilon_{\varphi(p)}(N)$ and the following diagram commutes.*

$$\begin{array}{ccc} \Upsilon_p(M) & \xrightarrow{\varphi_*} & \Upsilon_{\varphi(p)}(N) \\ \downarrow{\scriptstyle \exp_p} & & \downarrow{\scriptstyle \exp_{\varphi(p)}} \\ M & \xrightarrow{\varphi} & N \end{array}$$

In (d) the only requirement on the connection is that is should be preserved under diffeomorphisms, and that is the only reason why it should be the Riemannian connection. So all in all any connection that is preserved under diffeomorphisms has the property of the diagram. But in the following we need it specifically to be the Riemannian connection, or at least a connection that is compatible with the Riemannian metric, because we need geodesics to have constant speed.

Proposition 3.3. *Suppose that a group of isometries on a Riemannian manifold M acts transitively, then M is geodesically complete.*

A group acts transitively if there for all point $p, q \in M$ exists an element of the group, $\varphi : M \to M$, such that $\varphi(p) = q$.

Proof. Let $p \in M$ be given. Let $\varepsilon > 0$ such that the maximal geodesic exists for all startvectors $V \in T_pM$ where $|V| < \varepsilon$ (where $|V| = \sqrt{g_p(V,V)}$). Such an ε can be found, because exp is defined on an open set. These maximal geodesics are given by the exponential map defined on the tangent bundle. That there exists a maximal geodesic with startvectors for all $V \in T_pM$ where $|V| < \varepsilon$, means that $\exp_p V$ exists for all $|V| < \varepsilon$, when $V \in T_pM$. We denote the maximal geodesic $\gamma_V(t) = \exp_p(tV)$, and if we can show that γ_V can be extended to the all of $\mathbb{R}_+$ then we can also extend it to all of $\mathbb{R}_-$.

Assume that the maximal geodesic with $\gamma'(0) = V$ is only defined on (a, b) where $a < 0 < 1 < b < \infty$. Define $q = \gamma(b - \frac{1}{2})$ and define $w = \gamma'(b - \frac{1}{2})$. Now because we have a group of isometries that act transitively on M, we can find an isometry such that $\varphi : M \to M$ and $\varphi(p) = q$.

$|V| = \left|\gamma'(b - \frac{1}{2}\right| = |\gamma'(0)| < \varepsilon$, because Riemannian geodesics have constant speed. If we push W forward to T_pM with φ_*^{-1} then $\left|\varphi_*^{-1}(W)\right| < \varepsilon$ because φ is an isometry. So now we know that $\exp_p(\varphi_*^{-1}(W))$ exists. From the commutative diagram above we know that $\exp_q(W)$ exists, and so $\exp_q(sW) = \tilde{\gamma}(s)$ exists for $s \in [0; 1]$ and is a geodesic and $\tilde{\gamma}'(0) = W$. We are now able to extend γ to $(a, b + \frac{1}{2})$ by

$$\gamma(t) = \begin{cases} \gamma(t) & t \in (a, b) \\ \tilde{\gamma}(t - b + \frac{1}{2}) & t \in (b - \frac{1}{2}, b + \frac{1}{2}) \end{cases}$$

and $\tilde{\gamma}'(b - \frac{1}{2}) = W$ so the old γ and the new γ coincide, so the maximal geodesic was not so maximal after all, and that gives a contradiction. So a maximal geodesic can be defined on all of $\mathbb{R}_+$, and by the remark in the beginning of the proof, M is geodesically complete. □

4 The Gaussian curvature of the 2-dim hyperbolic space

In general, given a chart $(U, \underline{x}), \underline{x} = (x^1, \ldots, x^n)$ in a Riemannian manifold M, the Riemannian connection is defined by the Christoffel symbols as

$$\nabla_{\partial_i}\partial_j = \sum_k \Gamma^k_{ij}\partial_k,$$

where $\{\partial_i\} = \{\frac{\partial}{\partial x^i}\}$ is the standard familiy of vector fields that to each point in U gives a basis for the tangent space T_pM, and the Γ^k_{ij} are smooth functions, given by

$$\Gamma^k_{ij} = \frac{1}{2}\sum_l g^{il}(\partial_k g_{lj} + \partial_j g_{lk} - \partial_l g_{jk}), \tag{1}$$

where g_{ij} are the components of the Riemannian metric, and g^{ij} are the components of the inverse matrix.

From the Riemannian connection we can define the Riemannian curvature tensor Rm by

$$Rm(X,Y,Z,W) = g(R(X,Y)Z,W),$$

where X,Y,Z,W are vector fields on M, and R is the curvature tensor $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$.

By straight forward computations we can express functions for the curvature tensor in local coordinates by $R(\partial_i,\partial_j)\partial_l = \sum_k R^k_{lij}\partial_k$ where

$$R^k_{lij} = \partial_i\Gamma^k_{jl} - \partial_j\Gamma^k_{il} + \sum_h(\Gamma^h_{jl}\Gamma^k_{ih} - \Gamma^h_{il}\Gamma^k_{jh}). \tag{2}$$

Now we can express the Riemann curvature in the new R^k_{lij} as

$$R_{ijkl} = \sum_h g_{ih}R^h_{jkl}.$$

Furthermore we have the Ricci curvature tensor, Rc, given as the trace of the curvature tensor, so

$$R_{jl} = \sum_i R^i_{jil},$$

and the scalar curvature is the trace of the Ricci curvature

$$R = \sum_l g^{jl}R_{jl}.$$

By a corollary to Gauss' Theorema Egregium[1] we have that the Gaussian curvature, K, of a Riemannian 2-manifold is related to the curvature tensor, Ricci curvature tensor and scalar curvature by the formulas

$$\begin{aligned} Rm(X,Y,Z,W) &= K(g(X,W)g(Y,Z) - g(X,Z)g(Y,W)),\\ Rc(X,Y) &= K(g(X,Y)),\\ R &= 2K. \end{aligned} \tag{3}$$

With all these formulas we are now ready to calculate the Gaussian curvature of the Upper Half-plane.

We know that $g_{ij} = y^{-2}\delta_{ij}$ so $g^{ij} = y^2\delta^{ij}$. This is all we need to compute the Christoffel symbols and the Gaussian curvature. The symmetry of the Riemannian connection is expressed in local coordinates as commuting lower indices in the Christoffel symbols, so by (1)

$$\begin{aligned} \Gamma^1_{12} = \Gamma^1_{21} &= \frac{1}{2}y^2(\partial_1 g_{12} + \partial_2 g_{11} - \partial_1 g_{12})\\ &= \frac{1}{2}y^2\frac{\partial}{\partial y}y^{-2} = -y^{-1}. \end{aligned}$$

Similarly,

$$-\Gamma^2_{11} = \Gamma^2_{22} = -y^{-1}$$

and the rest are 0.

[1] Lemma 8.7 in [Lee(1997)]

By equation (2)

$$\begin{aligned} R^1_{212} &= -\partial_2 \Gamma^1_{21} + \partial_1 \Gamma^1_{22} + \sum_h (\Gamma^h_{22}\Gamma^1_{h1} - \Gamma^h_{21}\Gamma^1_{h2}) \\ &= -y^{-2} + 0 + (y^{-2} - y^{-2}) = -y^{-2}. \end{aligned}$$

Similarly,

$$\begin{aligned} R^2_{121} &= -y^2, \\ R^1_{111} &= R^2_{222} = 0, \\ R_{11} &= R^1_{111} + R^2_{121} = -y^{-2}, \\ R_{22} &= -y^{-2}, \\ R &= -2, \\ K &= -1. \end{aligned}$$

So the hyperbolic space in two dimensions has constant Gaussian curvature - 1.

Bibliography

[Lee(1997)] John M. Lee. *Riemannian Manifolds - An introduction to curvature*, volume 176 of *Graduate Texts in Mathematics*. Springer, 1997.