Computations of moduli spaces

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Contents

1 Introduction

The primary focus in this paper will be to calculate the $SU(2)$ moduli spaces of the simplest surfaces: The torus with zero and one puncture, and the sphere with three and four punctures. In the first chapter we define the moduli space of flat G-connections, where G is a Lie group, and discuss how flat connections are the same as representations of the fundamental group. We conclude the chapter with a view towards geometric quantization, and how moduli spaces fit perfectly into this picture.

Before we begin the actual calculations we discuss several nice results about the Lie group $SU(2)$ and its Lie algebra $\mathfrak{su}(2)$. Following this chapter we make a thoroughly investigation of the $SU(2)$ representation and character variety. In the last half of this paper, the different moduli spaces are calculated.

In this paper $\Sigma_{g,n}$ will be a compact orientable 2-dimensional manifold of genus g and with n boundary components, and G will be a Lie group.

2 Moduli space of flat G-connections

Let Σ be a closed Riemann surface of genus g. The moduli space of flat $G = SU(2)$ connections on Σ has two convenient descriptions. A topological, and a geometric. We could in fact use any Lie group, but we'll only be interested in the $SU(2)$ case.

In the geometric description we consider the space $\mathcal{A}_F$ of smooth flat connections on the trivial G-bundle $P \to \Sigma$. Given a fixed trivialization $P = G \times \Sigma$ of P, the space of connections on Σ, $\mathcal{A}$, can be identified with the space $\Omega^1 \otimes \mathfrak{g}$ of $\mathfrak{g}$-valued 1-forms on Σ, and $\mathcal{A}_F$ with the subset of $\mathcal{A}$ given by $\{A \in \mathcal{A} : F_A = dA + A \wedge A = 0\}$. The Gauge group $\mathcal{G} = \mathrm{Maps}(\Sigma, G)$ acts on $\mathcal{A}_F$ with a map $g \in \mathcal{G}$ taking $A \in \mathcal{A}_F$ to $A^g = g^{-1}Ag + g^{-1}dg$. Then the moduli space is defined as $\mathcal{M}(\Sigma, G) = \mathcal{A}_F/\mathcal{G}$.

The topological description of the moduli space is as the set of conjugacy classes of representations of the fundamental group $\pi_1(\Sigma)$ into G. This is the description we will use in the rest of this paper. In the remaining part of this section, we will look at the correspondence between these two descriptions, the so called holonomy map, and state some of the nice properties about these moduli spaces.

From connections to representations

A connection in a principal G-bundle can be seen as parallel transport between the fibers of the bundle. Given a curve $\gamma : [0,1] \to \Sigma$. The parallel transport will depend on γ and the connection, A. If A is flat the parallel transport only depend on the connection and the homotopy class of γ. If $\gamma(0) = \gamma(1)$ there will exists an element g relating the original element of $P_{\gamma(0)}$ with the transported element, by acting with g on the element. By fixing a flat connection A, the map assigning to each homotopy class of $\pi_1(\Sigma)$ an element of G is called the holonomy map, hol_A, and hol_A is a homomorphism. This gives a map $hol : \mathcal{A}_F \to \mathrm{Hom}(\pi_1(\Sigma), G)$. It can be shown that hol is also well-defined on the quotient $\mathcal{A}_F/\mathcal{G} \to \mathrm{Hom}(\pi_1(\Sigma), G)/G$.

From representations to connections

It is also possible to construct a principal G-bundle from a representation $\rho : \pi_1(\Sigma) \to G$. Let $\pi : \tilde{\Sigma} \to \Sigma$ be the universal cover of Σ. Fix a base point $p_0 \in \tilde{\Sigma}$ and identify $\tilde{\Sigma}$ with the space of homotopy classes of paths in Σ, starting at x_0, $x_0 = \pi(p_0)$. Then $\pi_1(\Sigma, x_0)$ naturally acts on $\tilde{\Sigma}$ from the right by concatenation of paths. For a representation $\rho : \pi_1(\Sigma, x_0) \to G$ we define the principal G-bundle to be the associated bundle to ρ, $P_\rho = \tilde{\Sigma} \times_\rho G = (\tilde{\Sigma} \times G)/\sim$, where $(\tilde{x}, g) \sim (\tilde{x} \cdot \alpha, \rho(\alpha)^{-1} g)$, $(\tilde{x}, g) \in \tilde{\Sigma} \times G$ and $\alpha \in \pi_1(\Sigma)$.

Pulling back the Maurer-Cartan form $\theta \in \Omega^1(G; \mathfrak{g})$ to $\tilde{\Sigma} \times G$ defines a natural flat connection $\tilde{A} = \pi_G^*(\theta)$ on $\tilde{\Sigma} \times G$, and since the Maurer-Cartan form is left-invariant this induces a flat connection on $\tilde{\Sigma}\times_\rho G$. It can be shown that this construction only depend on the conjugacy class of ρ, and is an inverse of the holonomy map, hol. Thus the holonomy map is a bijection $hol : \mathcal{M}(\Sigma, G) = \mathcal{A}_F \to \mathcal{G}to\mathrm{Hom}(\pi_1(\Sigma), G)/G$. All details can be found in [Him].

For later use, we introduce some terminology. $\mathrm{Hom}(\pi_1(\Sigma), G)$ are representations (provided G act on a vectorspace or a Hilbert space) and is called the *representation variety*. $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ is often given coordinates using the trace of a representation, and so this space is called the *character variety*, since the trace of a representation is the character of the representation.

Properties of the moduli space

Smoothness

As we will see in the calculations of in this paper, $\mathcal{M}(\Sigma, G)$ is in general not smooth, but a singular variety. It can however, be shown that the singularities are at most quadratic, [Gol1]. Furthermore it can be shown, that the irreducible representations $\mathcal{M}^{irr}(\Sigma, G) = \mathrm{Hom}^{irr}(\pi_1(\Sigma), G)/G \subset \mathcal{M}(\Sigma, G)$ constitute an open dense subset, and is a smooth manifold.

Another way to make the moduli space smooth, is to introduce a puncture on Σ, $\Sigma' = \Sigma \setminus \{pt\}$. Let γ be a loop around the puncture, and choose an element of a maximal torus of G (e.g. if $G = SU(2)$ pick $D = e^{i\theta}$, $\theta \in [-\pi, \pi]$) to be the corresponding element in G of γ. Then $\mathcal{M}_D(\Sigma', G) = \{\rho \in \mathrm{Hom}(\pi_1(\Sigma'), G) : \rho(\gamma) = D\}/G$ is a smooth submanifold of $\mathcal{M}(\Sigma', G)$.

Setup of geometric quantization

In [Gol1] Goldman defines a symplectic structure on both of the above smooth manifolds, by identifying tangent spaces with first cohomology of Σ with coefficients in the adjoint-bundle.

By the existence of a symplectic form, we could ask for the existence of a prequantum line bundle, i.e. an hermitian line bundle with a connection whose curvature is a scalar times the symplectic form. Several people (e.g. [Fre]) have produced these line bundles.

Furthermore Narasimhan and Seshadri ([NS]) tells us that the moduli spaces have Kähler structures, and Atiyah and Bott ([AB]) that (at least for $G = SU(n)$) $\mathcal{M}(\Sigma, G)$ is simply connected and $\mathrm{Im}(H^2(\mathcal{M}(\Sigma, G), \mathbb{Z}) \to H^2(\mathcal{M}(\Sigma, G), \mathbb{R}))$ is generated by $n[\omega]$, where ω is the symplectic form. Something similar is true when G is a compact Lie group.

This give all the ingredients for geometric quantization, since these ingredients are all what is needed to build a Hitchin connection in the Verlinde bundle over Teichmüller space. With this connection we will be able to connect the fibers in the Verlinde bundle, which is the quantum spaces comming from different choises of complex structure (i.e. from different polarizations).

We will not go further into the details of this story, but have included it to underline, why moduli spaces are important, and why you want to study them.

3 Basic properties of $SU(2)$

In this chapter we will discuss basic properties of the Lie group $SU(2)$ and its Lie algebra $\mathfrak{su}(2)$.

3.1 Conjugacy classes determined by trace

The conjugacy classes of $SU(2)$, are determined by the eigenvalues, since each element can be diagonalized by conjugation in $SU(2)$, and the eigenvalues are the diagonal entries. Eigenvalues lie on the circle of radius 1, as the determinant always is 1 in $SU(2)$.

The matrix trace gives a bijection between $SU(2)/\sim$, where $\sim$ is $SU(2)$-conjugation, and the interval $[-2, 2]$. Indeed, since the characteristic polynomial of a 2×2-matrix, X, have the form $P_X(\lambda) = \lambda^2 - \mathrm{Tr}(X)\lambda + \det(X)$, and determinants are always 1 in $SU(2)$, matricies with same trace have identical eigenvalues, hence Tr is injective. The image of Tr is $[-2, 2]$, since $\mathrm{Tr}(\begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix}) = 2\,\mathrm{Re}(\alpha)$, and eigenvalues being circle valued implies, $\mathrm{Re}(\alpha) \in [-1, 1]$.

3.2 Fiber of the matrix trace

Now that we have identified the quotient space $SU(2)/\sim$ with $[-2, 2]$, it would be interesting, to determine the fibers, i.e. to find the content of each conjugacy class.

If $A \in SU(2)$ then, $A = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$, where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$. The fiber over $t \in [-2, 2]$, are those matricies where $\mathrm{Re}(a) = \frac{t}{2}$. Inserting this, the determinant formula implies that the matricies should satisfy $\frac{t^2}{4} + \mathrm{Im}(a)^2 + |b|^2 = 1$, to be in the fibre over t. This is equivalent to $1 = \frac{\mathrm{Im}(a)^2}{c(t)} + \frac{|b|^2}{c(t)}$, where $c(t) = 1 - \frac{t^2}{4}$, or to

$$\left(\frac{\mathrm{Im}(a)}{\sqrt{c(t)}}\right)^2 + \left|\frac{b}{\sqrt{c(t)}}\right|^2 = 1, \tag{1}$$

if $t \neq \pm 2$.

Equation (1) is the equation for a 2-sphere: $S^2 = \{(z, t) \in \mathbb{C} \times \mathbb{R} \mid |z|^2 + t^2 = 1\}$.

If $t = 2$, $\mathrm{Re}(a) = 1$, which implies $a = 1$ and $b = 0$, so the fiber is just Id. Similarly if $t = -2$ the fibre is $-Id$.

3.3 S^3 and $SU(2)$ are Lie group isomorphic

In this section, we will show that S^3 given a group structure from the unit quaternions, is Lie group isomorphic to $SU(2)$.

The unit quaternions are elements of $\mathbb{H}$, $a+bi+cj+dk$, where $a^2+b^2+c^2+d^2 = 1$, $a, b, c, d \in \mathbb{R}$. The multiplication is ordinary multiplication, where we use the identities $i^2 = j^2 = k^2 = ijk = -1$. Since $w\overline{w} = 1$ for all unit quaternions, w, we define the group inverse to be is quaternion conjugation. These operations are clearly smooth, so S^3 is a Lie group.

The idea is to identify the unit quaternions and $SU(2)$, via a map from quaternions to 2×2-matricies.

There is an injection $\mathbb{C} \to \mathbb{H}$ by identifying the complex i with $i \in \mathbb{H}$, and a bijection $\mathbb{C}^2 \to \mathbb{H}$ by $(a + bi, c + di) \mapsto a + bi + cj + dk = (a + bi) + (c + di)j$. We can use this bijection to define an endomorphism of $\mathbb{C}^2$, by quaternionic multiplication with a given element, $\gamma = w_1 + jw_2 \in \mathbb{H}$. Expanding the multiplication, and using $wj = j\overline{w}$ for all $w \in \mathbb{C}$, we find

$$\begin{aligned}\gamma(z_1 + jz_2) &= (w_1 + jw_2)(z_1 + jz_2) = w_1z_1 + jw_2jz_2 + jw_2z_1 + w_1jz_2 \\ &= (w_1z_1 - \overline{w_2}z_2) + j(w_2z_1 + \overline{w_1}z_2).\end{aligned}$$

Quaternionic multiplication by γ is the same as matrix multiplication by $\varphi(\gamma) = \begin{pmatrix} w_1 & -\overline{w_2} \\ w_2 & \overline{w_1} \end{pmatrix}$, which defines a map $\varphi : \mathbb{H} \to \mathrm{Mat}_2(\mathbb{C})$.

Proposition 3.1. *$SU(2)$ is diffeomorphic to S^3 and Lie group isomoprhic to the unit quaternions.*

Proof. If $\gamma \in \mathbb{H}$ is a unit quaternion, the norm of γ is 1, and thus written as an element of $\mathbb{C}^2$, $1 = |\gamma|^2 = |w_1|^2 + |w_2|^2 = \det(\varphi(\gamma))$. Likewise its easy to see that $\varphi(\gamma)^*\varphi(\gamma) = Id$, and thus $\varphi(\gamma) \in SU(2)$.

This map is clearly smooth, since its components are linear functions of the coordinates. Its also clearly injective, so we need only to prove surjectivity. Surjectivity is given by the requirements $\det(A) = 1$ and $A^*A = Id$ for all elements in $A \in SU(2)$, since they imply that A has a very specific form, $A = \begin{pmatrix} w_1 & -\overline{w_2} \\ w_2 & \overline{w_1} \end{pmatrix}$, for some $w_1, w_2 \in \mathbb{C}$.

If φ should be a Lie group isomorphism, we only need it to be a group homomorphism. But this is obvious, since φ is defined by quaternionic multiplication. □

3.4 The Lie algebra $\mathfrak{su}(2)$

The Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ is by definition the tangent space if $SU(2)$ at 1. If we want to describe $\mathfrak{su}(2)$ in terms of matricies we can use the exponential map, $\exp : T_1SU(2) \to SU(2)$ by taking the exponential of an element. This is well-defined if you regard the tangent space as matricies or as quaternions.

If $e^\alpha \in SU(2)$ then $1 = \det e^\alpha = e^{\mathrm{Tr}(\alpha)}$, which happens if and only if $\mathrm{Tr}(\alpha) = 0$. Likewise should e^α satisfy $\overline{e^\alpha}^T = (e^\alpha)^{-1}$, which happens if and only if $\overline{\alpha}^T + \alpha = 0$. These restrictions implie that $\mathfrak{su}(2) = T_1SU(2)$ is the linear subspace of $Gl(2, \mathbb{C})$ consisting of traceless skew-hermitian matricies. These are all of the form $\begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix}$, $a \in \mathbb{R}$ and $b \in \mathbb{C}$. If we assume $b = u + iv$, $u, v \in \mathbb{R}$ then

$$\begin{pmatrix} ia & u+iv \\ -u+iv & -ia \end{pmatrix} = a\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + u\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + v\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = ai + uj + vk.$$

By the above calculation $\mathfrak{su}(2)$ is the purely imaginary quaternions, or $\mathbb{R}^3$.

Adjoint action on $\mathfrak{su}(2)$

We would like to investigate how the conjugation action of $SU(2)$ on itself, could be transfered to an action of $SU(2)$ on $\mathfrak{su}(2)$.

As we know, $SU(2)$ acts on itself by conjugation, the so called adjoint action, $A \mapsto Ad_A : SU(2) \to SU(2)$, $Ad_A(B) = ABA^{-1}$. For every A the derivative of the adjoint map at 1, d_1Ad_A, gives an action of $SU(2)$ on its tanget space at 1, i.e. on the Lie algebra, $\mathfrak{su}(2)$. This action is also denoted Ad_A,

$$A \mapsto Ad_A : \mathfrak{su}(2) \to \mathfrak{su}(2), \quad Ad_A(\alpha) = A\alpha A^{-1}.$$

Ad_A is a Lie algebra homomorphism since the Lie bracket on $\mathfrak{su}(2)$ is the commutator $[\alpha, \beta] = \alpha\beta - \beta\alpha$, so for every $A \in SU(2)$, $Ad_A \in Aut(\mathfrak{su}(2))$, and $Ad : SU(2) \to Aut(\mathfrak{su}(2))$ is a homomorphism, since $Ad_{AB} = Ad_AAd_B$, which is obvious from the definition of Ad_A.

Since $\mathfrak{su}(2) = \mathbb{R}^3$ is a linear space, automorphisms can be seen as linear maps. This extends the adjoint map to $Ad : SU(2) \to Gl(3, \mathbb{R})$. The scalar product in $\mathbb{R}^3 = \mathfrak{su}(2)$ kan be written in terms of matricies or quaternions as $u \cdot v = -\frac{1}{2}\operatorname{Tr}(uv) = -\operatorname{Re}(uv)$. An easy calculation shows that Ad_A preserves this scalar product

$$(Ad_A u) \cdot (Ad_A v) = -\frac{1}{2}\operatorname{Tr}(AuA^{-1}AvA^{-1}) = -\frac{1}{2}\operatorname{Tr}(AuvA^{-1}) = -\frac{1}{2}\operatorname{Tr}(uv) = u \cdot v,$$

and hence $Ad(SU(2)) \subset O(3)$. $SU(2)$ is connected and since Ad is continuous, the image of Ad will be in the same connected component as $1 \in O(3)$, hence $Ad(SU(2)) \subset SO(3)$.

We have proved the first part of the following proposition.

Proposition 3.2. *The adjoint action of $SU(2)$ on $\mathfrak{su}(2)$, is a well-defined Lie group homomorphism $SU(2) \to SO(3)$. This is the universal double cover of $SO(3)$, and hence $\pi_1 SO(3) = \mathbb{Z}_2$.*

The fact about the fundamental group of $SO(3)$, follows from general theory about the fundamental group, if we can show the second part of the proposition.

Proof. Let's consider surjectivity. Every matrix in $SO(3)$ can be viewed as a rotation around a coordiante axis in $\mathbb{R}^3$ – that is if we consider the elements in $SO(3)$ as linear transformations on $\mathbb{R}^3$. Therefore it is enough to show that we can hit each element of the form

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix}.$$

Rotation around the other two coordiante axes can be treated in the same manner.

Let $\varphi = \frac{\psi}{2}$ and $A = e^{i\varphi}$. Let's determine Ad_A in the basis i, j, k.

$$\begin{aligned} Ad_A(i) &= e^{i\varphi} i e^{-i\varphi} = i \\ Ad_A(j) &= e^{i\varphi} j e^{-i\varphi} = e^{2i\varphi} j = e^{i\psi} j = cos\psi j + \sin\psi k \\ Ad_A(k) &= e^{i\varphi} k e^{-i\varphi} = e^{2i\varphi} k = e^{i\psi} k = \cos\psi k - \sin\psi j \end{aligned}$$

so

$$Ad_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix},$$

which proves surjectivity.

Assume $Ad_A = Ad_B$, so $ACA^{-1} = Ad_A(C) = Ad_B(C) = BCB^{-1}$ for every $C \in SU(2)$. This implies that $B^{-1}A$ is in the center of $SU(2)$, hence $B^{-1}A = \pm 1$ or equivalently $A = \pm B$. In other words $Ad : SU(2) \to SO(3)$ is $2-1$. □

Proposition 3.3. *The exponential map gives a diffeomorphism*

$$\exp : B_\pi(0) \to SU(2) \setminus \{-1\}$$

between the ball of radius π centered at the origin $B_\pi(0) \subset \mathfrak{su}(2) = \mathbb{R}^3$, and $SU(2)$ less -1.

Proof. Any $SU(2)$-matrix can be diagonalized, so given $A \in SU(2)$, there exists a $g \in SU(2)$ such that $gAg^{-1} \in S^1$. Assume $A \neq -1$, then if $\exp : (-i\pi, i\pi) \to S^1 \setminus \{-1\}$ is a diffeomorphism, there exists a $\beta \in (-i\pi, i\pi)$ such that $e^\beta = gAg^{-1}$, which is equivalent to $A = g^{-1}e^\beta g = e^{Ad_{g^{-1}}\beta}$. Since $Ad_{g^{-1}}$ is a rotation $Ad_{g^{-1}}(\beta) \in B_\pi(0)$, and $\exp : B_\pi(0) \to SU(2) \setminus \{-1\}$ is surjective.

If $\beta_1, \beta_2 \in B_\pi(0)$ and $e^{\beta_1} = e^{\beta_2}$, this is equivalent to $e^{\beta_1 - \beta_2} = 1 \in S^1 \setminus \{-1\}$ where exp was a diffeomorpshims, so $\beta_1 - \beta_2 = 0$, and $\exp : B_\pi(0) \to SU(2) \setminus \{-1\}$ is injective. The only thing we need is $\exp : (-i\pi, i\pi) \to S^1 \setminus \{-1\}$ being a diffeomorphism, which is obvious. $\square$

4 The representation variety

4.1 Its topology

Let π be a group given by generators and relations as $\pi = \langle f_1, \dots, f_n \mid R_1, \dots R_m \rangle$, this description is carried to the space of homomorphisms, $\mathrm{Hom}(\pi, G) = \{G^n | R_1(f_1, \dots, f_n) = 1, \dots R_m(f_1, \dots, f_m) = 1\}$. This space is also called the representation variety. If G is a linear Lie group, which it will be in our case, it's clearly a real algebraic set, since the relations together with the defining relations for G, gives a set of polynomial equations that carve out $\mathrm{Hom}(\pi, G)$ inside $\mathbb{R}^N$. It can be shown that $\mathrm{Hom}(\pi, G)$ is actually a real affine variety. Assume, that G is a linear Lie group.

By identifying the space of homomorphisms with an affine variety, it becomes a topological space – in several ways. As a variety it has the Zariski topology, but it also has the induced topology from the product topology on G^n, and lastly it has the compact open topology. At first sight these topologies seems to depend on the choice of generators and relations. Since one choice is not preferred over another, we should show that the topology is independent of the generators and relations of π.

Assume that $G_2 = \left\langle h_1, \dots, h_k \mid \tilde{R}_1, \dots \tilde{R}_l \right\rangle \simeq \pi \simeq \langle g_1, \dots, g_n \mid R_1, \dots, R_m \rangle = G_1$. Define F and H as the identifications $F : G_1 \xrightarrow{\simeq} \pi \xrightarrow{\simeq} G_2$ and $H : G_2 \xrightarrow{\simeq} \pi \xrightarrow{\simeq} G_1$. By definition $F \circ H = H \circ F = Id$, and being group homomorphisms we only need to define them on the set of generators, $F(g_i) = w_i(h_1, \dots, h_k)$, $H(h_j) = z_j(g_1, \dots, g_n)$ where the w_i are words in h_j's and z_j's are words in g_i's. These define pullbacks on the space of homomorphisms.

$F^* : \mathrm{Hom}(G_2, G) \to \mathrm{Hom}(G_1, G)$, by $F^*(\rho)(g) = \rho(F(g))$ and likewise for $H^* : \mathrm{Hom}(G_1, G) \to \mathrm{Hom}(G_2, G)$. $F^*(\rho)(g_i) = \rho(F(g_i)) = \rho(w_i(h_1, \dots, h_k)) = w_i(\rho(h_1), \dots, \rho(h_k))$, so by abusing notation, $F = (w_1, \dots, w_k)$. In the same way $H = (z_1, \dots, z_n)$.

From this it's clear that F^* and H^* are inverses of each other, and that the two spaces of homomorphisms are in bijection. The only thing remaining is F^* and H^* beeing continuous in the different topologies. If we equip $\mathrm{Hom}(\pi, G)$ with the induced topology, each of the w_i, z_j's are continuous since the group operations are continuous and w_i's and z_j's are just monomials. The same kind of reason makes F^* and H^* continuous if $\mathrm{Hom}(\pi, G)$ is equipped with the Zariski topology. F^* an H^* are both defined by polynomials in the coordinates on G^n and G^k respectively, and hence are regular maps of the varieties. Since regular maps are continuous in the Zariski topology, F^* and H^* are continuous.

Any of the two topologies put on the space of homomorphisms, is well-defined.

We can give $\mathrm{Hom}(\pi, G)$ a third topology, the compact-open topology, where π is given discrete topology, and G the induced topology from $\mathbb{R}^N$. A basis for this topology, consists of sets $U_{K,V} = \{f : \pi \to G \mid f(K) \subset V\}$ for all compact $K \subset \pi$ and all open $V \subset G$. Note that a set is compact in the discrete topology exactly if it is finite. Therefore, the topology on $\mathrm{Hom}(\pi, G)$ can be described by its sub-base, which consists of all subsets $U_{g,V} = \{f : \pi \to G \mid f(g) \in V\}$ of $\mathrm{Hom}(\pi, G)$, one for each $g \in \pi$ and each open subset $V \subset G$. By comparing this sub-base with the sub-base from the induced product-topology given by the presentation of π, it can be shown, that the two topologies are equal.

In the following π will be the fundamental group of a compact surface, and $G = SU(2)$. Since $SU(2)$ is compact, the relations from π will define a closed subset of $SU(2)^n$, where n is the number of relations on a presentation of π and hence $\mathrm{Hom}(\pi, SU(2))$ will be compact. Note that given the Zariski topology real affine varieties are always compact.

4.2 Irreducible representations

A representation $\rho : \pi \to G$ is called *irreducible* if its stabilizer $S_\rho = \{g \mid g\rho g^{-1} = \rho\}$ coincides with the center $C_G = \{h \,|\, gh = hg \text{ for all } g \in G\}$ of G. In our case $C_{SU(2)} = \{\pm 1\}$. If the stabilizer contains more elements than the center, the representation is called *reducible*.

There are many different ways of defining (ir)reducibility of representations, and the following proposition shows that they are all equivalent.

Proposition 4.1. *Let $\rho : \pi \to SU(2)$ be a representation of the group π on $\mathbb{C}^2$. The following are equivalent*

(i) ρ *is reducible*

(ii) $\mathrm{Im}(\rho) \subset U(1) \subset SU(2)$, *i.e. ρ factors through a copy of $U(1)$ in $SU(2)$.*

(iii) There exists a non-zero $\mathbb{C}$-linear proper subspace, U, of $\mathbb{C}^2$, invariant under ρ, i.e. $\rho(g)(U) \subset U$ for all $g \in \pi$.

Proof. *(iii)* $\Rightarrow$ *(ii)*: Assume there exists a subspace of $\mathbb{C}^2$, which is invariant under ρ. In a suitable basis for $\mathbb{C}^2$ $\rho(g) = \begin{pmatrix} a_g & b_g \\ 0 & c_g \end{pmatrix}$, where $g \in \pi$. Since $\rho(g) \in SU(2)$, $b_g = 0$ and $|a_g|^2 = 1$, and $\rho(g) = \begin{pmatrix} e^{i\theta_g} & 0 \\ 0 & e^{-i\theta_g} \end{pmatrix}$ for a suitable $\theta_g \in [0, 2\pi]$, and $\mathrm{Im}(\rho) \subset U(1) \subset SU(2)$.

(ii) $\Rightarrow$ *(iii)*: If $\rho : \pi \to SU(2)$ factors through a copy of $U(1)$ in $SU(2)$, it is clear that you can find a proper subspace of $\mathbb{C}^2$ invariant under ρ.

(ii) $\Rightarrow$ *(i)*: If $\rho : \pi \to SU(2)$ factors through a copy of $U(1)$ in $SU(2)$, it is clear that the stabilizer of ρ is more than the center of $SU(2)$. Since the image of ρ is a torus, the image will be included in the stabilizer, and the representation is reducible.

(i) $\Rightarrow$ *(ii)*: Let $\beta \in S_\rho \setminus Z(SU(2))$. β acts on $SU(2)$ and choose a basis for $\mathbb{C}^2$ such that $\beta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$. $\rho(g) = \begin{pmatrix} \alpha_g & \overline{\gamma}_g \\ -\gamma_g & \overline{\alpha}_g \end{pmatrix}$, where $|\alpha_g|^2 + |\gamma_g|^2 = 1$. By definition of β, $\beta\alpha(g) = \alpha(g)\beta$, and

$$\begin{aligned} \rho(g)\beta \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= e^{i\theta} \begin{pmatrix} \alpha_g \\ -\gamma_g \end{pmatrix} \\ \beta\rho(g) \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \beta \begin{pmatrix} \alpha_g \\ -\gamma_g \end{pmatrix} = \begin{pmatrix} e^{i\theta}\alpha_g \\ -e^{-i\theta}\gamma_g \end{pmatrix} \end{aligned}$$

Since $\beta\rho(g) = \rho(g)\beta$, $e^{i\theta}\gamma_g = e^{-i\theta}\gamma_g$, and since $\beta \neq \pm 1$ $\theta \neq 0, \pi$, so $\gamma_g = 0$. In other words the image of ρ is contained in a copy of $U(1)$ in $SU(2)$. □

Suppose $\rho : \pi \to SU(2)$ is reducible, then the image will be contained in a copy of $U(1)$ in $SU(2)$. If we assume π has n generators, we can view ρ as an element of $U(1)^n \subset SU(2)^n$, that is $\rho \in \mathrm{Hom}(\pi, SU(2)) \cap U(1)^n$. Conversely, every element of $\mathrm{Hom}(\pi, SU(2)) \cap U(1)^n$ is a reducible representation, so $\mathrm{Hom}(\pi, SU(2)) \cap U(1)^n$ is all reducible represntations of π in $SU(2)$. This set is closed, since $U(1)$ is closed in $SU(2)$. This implies that the irreducible elements of $Rep(\pi) = \mathrm{Hom}(\pi, SU(2))$, $Rep^{irr}(\pi)$ is an open subset.

Suppose $\rho : \pi \to SU(2)$ is irreducible, then $S_\rho = \{\pm 1\}$, the center of $SU(2)$. The adjoint action of $SU(2)$ on $Rep(\pi)$ is not free, since every element has a stabilizer, which is more than just the identity. But if we restrict our attention to the action of $SO(3) = SU(2)/\{\pm 1\}$ on $Rep^{irr}(\pi)$, then it's free. If $h, g \in SU(2)$ and $g = -h$, then $gSU(2)g^{-1} = hSU(2)h^{-1}$, and $Rep^{irr}(\pi)/SU(2)$ consists of the same orbits as $Rep^{irr}(\pi)/SO(3)$. The difference is that $Rep^{irr}(\pi)/SU(2)$ can contain singularities, but they disapear if our action is free, as it is in the case of $SO(3)$ acting on $Rep^{irr}(\pi)$. The character variety is $\mathcal{M}(\pi) = Rep(\pi)/SU(2)$ but to find the dimension, look at $Rep^{irr}(\pi)/SO(3)$, which is a smooth manifold – in a way, we're only interested in the nice part of the character variety.

4.3 Representations of free groups

Assume π is a free group of rank g, e.g. $\pi = \pi_1(M)$, where M is a handlebody of genus g. Then $Rep(\pi) = SU(2)^g$ and $\dim Rep(\pi) = 3g$. $Rep^{irr}(\pi) \subset Rep(\pi)$ is open, so $\dim Rep^{irr} = 3g$. Since $\dim SO(3) = \dim SU(2) = 3$, then $\dim Rep^{irr}(\pi)/SO(3) = 3g - 3$ and $Rep^{irr}(\pi)/SO(3)$ is a smooth open manifold, since the action is free. Then the dimension of $\mathcal{M}(\pi)$, is also $3g - 3$.

4.4 Representations of surface groups

In this section we concentrate on the character variety of surface groups, that is of the groups $\pi_1(\Sigma)$ where Σ is a compact Riemann surface. This section is based on [Sav].

Let Σ be a closed compact Rimeann surface of genus g, and $\Sigma_0 = \Sigma \setminus D^2$, where D^2 is an open disk in Σ. Let $\gamma = \partial\Sigma_0$ be the oriented boundary of Σ_0.

$\pi_1(\Sigma_0)$ is a free group of rank $2g$ with generators $a_1, b_1, \ldots, a_g, b_g$, and in this presentation $\gamma = \Pi_{n=1}^{g}[a_n, b_n] = \Pi_{n=1}^{g} a_n b_n a_n^{-1} b_n^{-1}$. $\pi_1(\Sigma)$ has the same $2g$ generators and a single relation $\gamma = 1$, that is the product of commutators should be a contractible curve.

Let $h : Rep(\pi_1(\Sigma_0)) \to SU(2)$ be the map evaluating a representation, ρ, on γ, i.e. $\rho \mapsto \rho(\gamma)$. By the above description of $Rep(\pi_1(\Sigma))$ it is clear that $Rep(\pi_1(\Sigma)) = h^{-1}(1)$. $Rep(\pi_1(\Sigma_0))$ is identified with $SU(2)^{2g}$ by the map sending a representation ρ to $(A_1, B_1, \ldots, A_g, B_g)$ where $A_i = \rho(a_i)$, $B_i = \rho(b_i)$. Therefore h is then $(A_1, B_1, \ldots, A_g, B_g) \mapsto \Pi_{n=1}^{g} A_n B_n A_n^{-1} B_n^{-1}$.

Theorem 4.2. *The map h is surjective. Furthermore, h is regular at irreducible representations, and only there. In other words, $d_\rho h$ is surjective if and only if ρ is irreducible*

Corollary 4.3. $\mathcal{M}(\pi(\Sigma))$ *has dimension* $6g - 6$.

With the theorem given, this is rather easy, since $\dim Rep(\pi_1(\Sigma_0)) = 6g$ and the regularity of h at irreducible representations implies that $Rep^{irr}(\pi_1(\Sigma))/SO(3)$ has dimension $6g - 3 - 3 = 6g - 6$, where one -3 is from $Rep(\pi_1(\Sigma)) = h^{-1}(1)$ and the constant rank level set theorem tells this dimension should be $6g - 3$, and the remaning -3 is from the quoutient with $SO(3)$.

Proof. Let $R_\varphi = e^{i\varphi}$, $h(\mathbb{R}_\varphi, j, 1, \ldots, 1) = e^{i\varphi} j e^{-i\varphi} j^{-1} = e^{2i\varphi} = R_{2\varphi}$. If $A \in SU(2)$ it can be diagonalized, by a $C \in SU(2)$, $A = CR_\varphi C^{-1}$ for a suitable φ. Then

$h(CR_{\varphi/2}C^{-1}, CjC^{-1}, 1, \dots, 1) = Ce^{i\varphi/2}je^{-i\varphi/2}j^{-1}C^{-1} = Ce^{i\varphi}C^{-1} = A$, and h is surjective.

The hard part of the theorem is to show regularity.

For every $\rho = (A_1, B_1, \dots, A_g, B_g)$ we have a natural identification of tangent spaces $T_\rho Rep(\pi_1(\Sigma_0)) = T_{A_1}SU(2) \oplus T_{B_1}SU(2) \oplus \cdots \oplus T_{A_g}SU(2) \oplus T_{B_g}SU(2)$. By using suitable left transformations we get an isomorphism

$$T_1SU(2) \oplus \cdots \oplus T_1SU(2) \xrightarrow[\simeq]{L} T_{A_1}SU(2) \oplus \cdots \oplus T_{B_g}SU(2),$$

where $L = L_{A_1} \oplus \cdots \oplus L_{B_g}$. In the same way, a right translation by $h(\rho)$ will give an identification of $T_1SU(2)$ with $T_{h(\rho)}SU(2)$. In the following diagram is the map D constructed in such a way that the diagram commutes.

$$\begin{tikzcd} T_{A_1}SU(2) \oplus \cdots \oplus T_{B_g}SU(2) \arrow[r, "d_\rho h"] & T_{h(\rho)}SU(2) \\ T_1SU(2) \oplus \cdots \oplus T_1SU(2) \arrow[u, "L"] \arrow[r, "D"] & T_1SU(2) \arrow[u, "R_{h(\rho)}"'] \end{tikzcd}$$

To show that $d_\rho h$ is surjective for irreducible ρ and only there, we can equivalently show that D is surjective if and only if ρ is irreducible.

Let $u_n \in T_1SU(2)$ in the tangent space corresponding to $T_{A_n}SU(2)$. Then $L_{A_n}(1+\varepsilon u_n) = A_n(1+\varepsilon u_n)$ and

$$\begin{aligned} h(A_1, \dots, &A_n(1+\varepsilon u_n), B_n, \dots, B_g) = \\ &= [A_1, B_1]\cdots[A_{n-1}, B_{n-1}][A_n(1+\varepsilon u_n), B_n][A_{n+1}, B_{n+1}]\cdots[A_g, B_g] \\ &= C_{n-1}[A_n(1+\varepsilon u_n), B_n]C_n^{-1}C_g, \end{aligned} \tag{2}$$

where $C_k = \Pi_{n=1}^k [A_n, B_n]$, and $C_0 = 1$, $C_g = h(\rho)$.

Let's try to calculate the n'th factor,

$$\begin{aligned} [A_n(1+\varepsilon u_n), B_n] &= A_n(1+\varepsilon u_n)B_n(1-\varepsilon u_n)A_n^{-1}B_n^{-1} + \mathcal{O}(\varepsilon^2) \\ &= [A_n, B_n] + \varepsilon(A_nu_nB_nA_n^{-1}B_n^{-1} - A_nB_nu_nA_n^{-1}B_n^{-1}) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

To calculate D in the point $(0, \dots, 0, u_n, 0, \dots, 0)$ we differentiate (2) with respect to ε, evauluate at $\varepsilon = 0$ and multiply by $h(\rho)^{-1}$ to return to $T_1SU(2)$.

$$\begin{aligned} D(0, &\dots, 0, u_n, 0, \dots, 0) = \\ &= R_{h(\rho)^{-1}}d_\rho h(L_{A_1} \oplus \cdots \oplus L_{B_g})(0, \dots, 0, u_n, 0, \dots, 0) \\ &= R_{h(\rho)^{-1}}(d_\rho h)(A_1, \dots, B_{n-1}, A_n(1+\varepsilon u_n), B_n, \dots, B_g) \\ &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}(h(A_1, \dots, A_n(1+\varepsilon u_n), B_n, \dots, B_g))h(\rho)^{-1} \\ &= C_{n-1}A_nu_nB_nA_n^{-1}B_n^{-1}C_n^{-1} - C_{n-1}A_nB_nu_nA_n^{-1}B_n^{-1}C_n^{-1} \\ &= C_{n-1}A_nu_nA_n^{-1}C_{n-1}^{-1} - C_{n-1}A_nB_nu_nB_n^{-1}A_n^{-1}C_{n-1}^{-1} \\ &= x_n - C_{n-1}A_nB_nA_n^{-1}C_{n-1}^{-1}x_nC_{n-1}A_nB_n^{-1}A_n^{-1}C_{n-1}^{-1}, \end{aligned}$$

since $h(\rho) = C_g$, $C_n^{-1} = [A_n, B_n]^{-1}C_{n-1}^{-1} = B_nA_nB_n^{-1}A_n^{-1}C_{n-1}^{-1}$ and $x_n = C_{n-1}A_nu_nA_n^{-1}C_{n-1}^{-1}$.

Let v_n be a vector in the tangent space $T_1SU(2)$ corresponding to $T_{B_n}SU(2)$. In exactly the same way as above, we show that

$$\begin{aligned} h(A_1, \dots, &A_n, B_n(1+\varepsilon v_n), \dots, B_g) = \\ &= [A_n, B_n] + \varepsilon(A_nB_nv_nA_n^{-1}B_n^{-1} - A_nB_nA_n^{-1}v_nB_n^{-1}) + \mathcal{O}(\varepsilon^2), \end{aligned}$$

and

$$\begin{aligned}
D(0,\dots,&0,v_n,0,\dots,0) = \\
&= R_{h(\rho)^{-1}}(d_\rho h)(L_{A_1}\oplus\cdots\oplus L_{B_g})(0,\dots,0,v_n,0,\dots,0)\\
&= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}(C_{n-1}[A_n,B_n(1+\varepsilon v_n)]C_n^{-1}C_g)C_g^{-1}\\
&= C_{n-1}A_nB_nv_nA_n^{-1}B_n^{-1}C_n^{-1} - C_{n-1}A_nB_nA_n^{-1}v_nB_n^{-1}C_n^{-1}\\
&= C_{n-1}A_nB_nv_nB_n^{-1}A_n^{-1}C_{n-1}^{-1} - C_{n-1}A_nB_nA_n^{-1}v_nA_nB_n^{-1}A_n^{-1}C_{n-1}^{-1}\\
&= y_n - C_{n-1}A_nB_nA_n^{-1}B_n^{-1}A_n^{-1}C_{n-1}^{-1}y_nC_{n-1}A_nB_nA_nB_n^{-1}A_n^{-1}C_{n-1}^{-1},
\end{aligned}$$

where $y_n = C_{n-1}A_nB_nv_nB_n^{-1}A_n^{-1}C_{n-1}^{-1}$.

The above calculations show that the image of D is the set of elements in $T_1SU(2)$, which can be written in the following manner

$$\sum_{n=1}^{g}(1-Ad_{F_n})x_n + \sum_{n=1}^{g}(1-Ad_{G_n})y_n,$$

where $x_n, y_n \in T_1SU(2)$ and $F_n = C_{n-1}A_nB_nA_n^{-1}C_{n-1}^{-1}$, $G_n = C_{n-1}A_nB_nA_n^{-1}B_n^{-1}A_n^{-1}C_{n-1}^{-1}$.

It can be shown that the set of elements $\{c_{n-1}a_nb_na_n^{-1}c_{n-1}^{-1}, c_{n-1}a_nb_na_n^{-1}b_n^{-1}a_n^{-1}c_{n-1}^{-1}\}_{n=1,\dots,g}$, where $c_k = \Pi_{n=1}^{k}[a_n,b_n]$ is a basis for the free group $\pi_1(\Sigma_0)$. As an example let's show that the generators for $n=1$ will give a_1 and b_1 by appropriate multiplications.

For $n=1$ the generators are $a_1b_1a_1^{-1}$ and $a_1b_1a_1^{-1}b_1^{-1}a_1^{-1}$. Then $(a_1b_1a_1^{-1})^{-1}a_1b_1a_1^{-1}b_1^{-1}a_1^{-1}(a_1b_1a_1^{-1}) = a_1^{-1}$, and so also b_1 is a word in the two generators. The same procedure can be applied to higher n, but it is more cumbersome.

With this new presentation of $\pi_1(\Sigma_0)$ a representation $(A_1,B_1,\dots,A_g,B_g)$ is reducible if and only if $(F_1,G_1,\dots,F_g,G_g)$ is reducible. We have to show that $d_\rho h$ is surjective precisely when ρ is irreducible, which is equivalent to $\sum_{n=1}^{g}(1-Ad_{F_n}) + \sum_{n=1}^{g}(1-Ad_{G_n})$ beeing surjective.

Let $F\in SU(2)$ and let's calculate the image of $1-Ad_F$. Assume $F\neq\pm1$ otherwise the image is 0. As in the Subsection 3.4 about the adjoint action of $SU(2)$ on the Lie algebra $T_1SU(2)=\mathfrak{su}(2)$, Ad_F is a non-trivial element in $SO(3)$, which is a non-trivial rotation with angle φ about some axis $\mathbb{R}_F$. Let $\mathbb{C}_F$ be the plane orthogonal to $\mathbb{R}_F$ such that $\mathbb{R}^3 = \mathbb{R}_F\oplus\mathbb{C}_F$. $\mathbb{R}_F$ is fixed by Ad_F and hence $\mathrm{Im}(1-Ad_F)=\mathbb{C}_F$.

Note that Ad_F and Ad_G commute if and only if F and G commute. If Ad_F and Ad_G are non-trivial rotations they commute if and only if the axes of rotation are equal, $\mathbb{R}_F=\mathbb{R}_G$.

Assume ρ is reducible, then by Proposition 4.1 $\rho(\pi_1(\Sigma_0))$ is in a copy of $U(1)$ in $SU(2)$, hence all F_n and G_n commute, and hence all Ad_{F_n} and Ad_{G_n} have the same axes of rotation, $\mathbb{R}_{F_1}$. Then $\mathrm{Im}\,D = \mathbb{C}_{F_1}\neq\mathbb{R}^3=\mathfrak{su}(2)=T_1SU(2)$.

If ρ is irreducible at least two of the operators Ad_{F_n}, Ad_{G_n} have different axes of rotation, $\mathbb{R}_1$, $\mathbb{R}_2$, and the image of D contains both $\mathbb{C}_1$ and $\mathbb{C}_2$, which spans $\mathbb{R}^3$. $\square$

5 The first moduli space

5.1 Moduli space of a torus

The fundamental group of a torus is one of the first you learn to calculate, in an introductory course on algebraic topology, and is $\pi_1(S^1\times S^1)=\mathbb{Z}\times\mathbb{Z}$. It is also given by generators and relations by $\pi_1(S^1 \times S^1) = \langle A, B \mid AB = BA \rangle$. Therefore the space of homomorphisms is $\mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}, SU(2)) = \{(A, B) \in SU(2) \times SU(2) \mid AB = BA\}$.

The moduli space is the quotient of the action of simultanious conjugation in each factor, so $g \cdot (A, B) = (gAg^{-1}, gBg^{-1})$. We can always conjugate A to be a diagonal matrix, with eigenvalues on the diagonal. Let's for now assume that A is not ± 1. Then $A = e^{2\pi i t}$, where $t \in (0, 1) \setminus \{\frac{1}{2}\}$. Since A and B commute, they also commute after conjugation. Therefore B is also a diagonal matrix. When $t = 0, \frac{1}{2}, 1$, B can be conjugated by all elements of $SU(2)$, since it doesn't change A, and we can again conjugate B to be a diagonal matrix. This gives a map

$$\begin{aligned} h : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} = \mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1 &\to \mathcal{M}(\Sigma_{1,0}) \\ (t, s) &\mapsto \left(e^{2\pi i t}, e^{2\pi i s}\right) \end{aligned}$$

This map is by construction surjective, and is clearly not injective, since (a, b) and $(-a, -b)$ is sent to two matricies with the same eigenvalues, and hence they are in the same conjugacy class. This ammounts to the fact that after diagonalizing A and B, we can conjugate further with $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, to switch the diagonal entries of A and B. Switching the eigenvalues is the only non-trivial conjugation we can do to the diagonalized matricies. If we qoutient $\mathbb{R}^2/\mathbb{Z}^2$ by the $\mathbb{Z}_2$ action identifing (a, b) and $(-a, -b)$, h will induce a bijection on the qoutient.

The $\mathbb{Z}_2$ action has four fixpoints: $(0, 0)$, $(0, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0)$, and on Figure 1 the identified lines are depicted. The fixpoints will be non-manifold points, and the rest will be a sphere. You can visualize this as a sphere with four spikes – or a pillow case if you like.

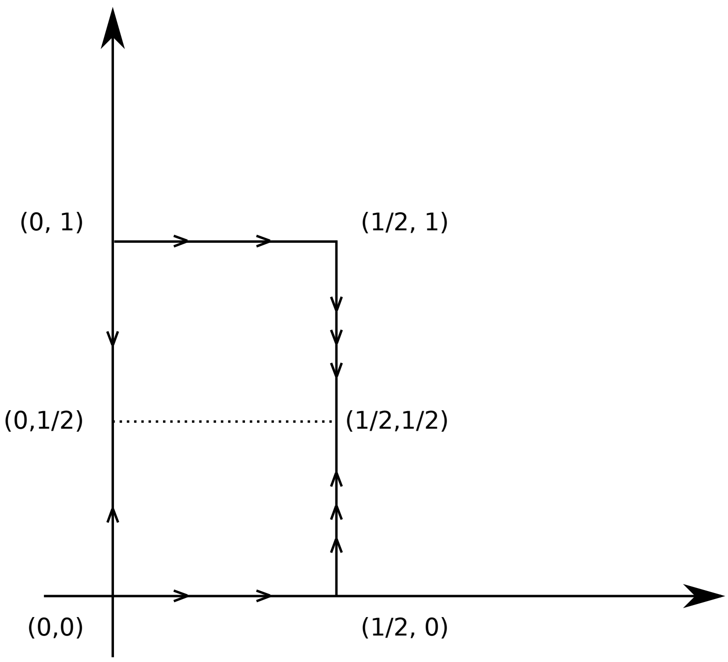


Figure 1: Quotient of the torus with the $\mathbb{Z}_2$ action

From the general theory of moduli spaces, we know that the irreducibel representations in $\mathrm{Hom}(\pi_1(\Sigma), SU(2))$ constitute a $6g - 6$ dimensional subvariety of the character variety, $M(\Sigma)$. In this case $g = 1$ and there should be 0-dimensional set of irreducible represenations in $\mathrm{Hom}(\pi_1(\Sigma), SU(2))/SU(2)$, but actually we already know, that there are none. $\pi_1(\Sigma_{1,0}) = \mathbb{Z} \times \mathbb{Z}$ is abelian, and hence all the irreducible representations should be 1-dimensional, and since we look at 2-dimensional represenations, there are none irreducible represenations to be found in $\mathrm{Hom}(\pi_1(\Sigma), SU(2))/SU(2)$.

6 Moduli space of a pair-of-pants

6.1 Pair-of-pants decomposition

If you want to study properties of a given object, it can be very fruitfull to look at small building blocks, constituting the object. It is a very common thing to do, if you want to build invariants of an object. Cut the object into small pieces, and define the invariant from these pieces. This is a common procedure in quantum topology, e.g. Turaev-Viro-invariants [TV] of 3-manifolds and Turaevs shadow invariant of 4-manifolds [Tur], just to name a few.

Our object is a compact surface, and a well known decomposition of a surface is a triangulation. Triangulations of compact surfaces can be derived from the fundamental polygons. Another useful decomposition is the "pair-of-pants" or trinion decomposition, it goes by many names, but the first seems to be used in the math community, while the last is more widely used in physics. Basically it is a decomposition of the surface into small pieces, each of which is a three-holed sphere. See figure 2 for an example. The red lines indicate, where the surface should be cut.

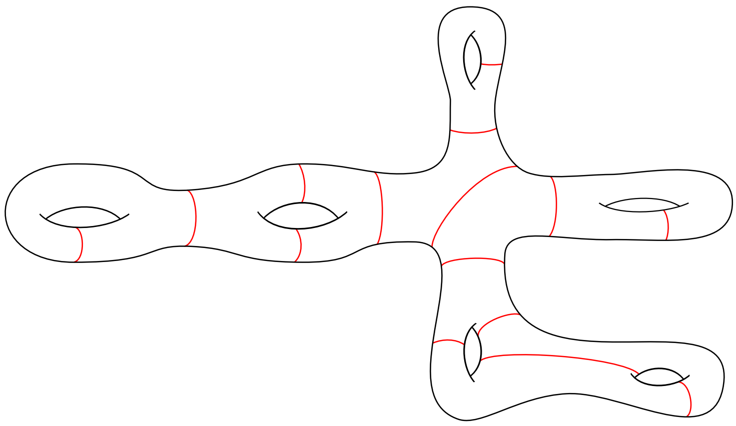


Figure 2: A pair-of-pants decomposition of $\Sigma_{6,0}$

In this section we will have a close look at pair-of-pants decompositions. We will prove that such a decomposition always exists (with very few exceptions), and given a compact surface $\Sigma_{g,n}$ of genus g and n boundary components, how many pairs of pants we get.

Definition 6.1. Let $\Sigma_{g,n}$ be a connected compact orientable surface of genus g, with n boundary components. We call this a surface of type (g, n). By a *pair-of-pants* decomposition of $\Sigma_{g,n}$, we mean a finite collection of disjoint smoothly embedded circles cutting $\Sigma_{g,n}$ into pieces, each of which are surfaces of type $(0, 3)$.

The following proposition could be used as a defining property for genus of a surface, however we define the genus of a surface as the number of handles you have to glue onto a sphere to obtain something, which is homeomorphic to your surface.

Proposition 6.2. *Let Σ_g be a compact connected orientable surface of genus g. There exists g disjoint circles in Σ_g, whose complement is connected, but any $g+1$ disjont circles disconnect Σ_g.*

Proof. If $g = 0$ we may assume $\Sigma_0 = S^2$. This is Jordans Curve Theorem.

Suppose $\gamma_1, \dots, \gamma_q$ are disjoint circles in Σ_g, $q \geq 1$, and $\Sigma_g \setminus \cup_{i=1}^q \gamma_i$ is connected. Let $N_1, \dots, N_q$ be disjoint closed tubular neighborhoods of $\gamma_1, \dots, \gamma_q$, and define $V = \Sigma_g \setminus \cup_{i=1}^q N_i$. Let W be obtaind from V by capping the $2q$ newly created boundary circles of V with disks. Notice that W is connected and orientable, and Σ_g is obtaind from W by attaching q handles. Let W have genus $p \geq 0$, then Σ_g has genus $g = p + q$, so $q \leq g$. □

Theorem 6.3. *If a compact connected orientable surface, $\Sigma_{g,n}$, of type (g, n), is not of type $(0,0)$, $(0,1)$, $(0,2)$ or $(1,0)$, there exists a finite set $\{\gamma_i\}_{i=1}^h$ of pairwise disjoint smoothly embedded circles on $\Sigma_{g,n}$, such that*

$$\Sigma_{g,n} \setminus \cup_{i=1}^h \gamma_i$$

consists of k connected components, each of which has closure in $\Sigma_{g,n}$ of type $(0,3)$.

The numbers h and k are uniquely determined by the type (g, n)

$$h = 3g - 3 + n \qquad k = 2g - 2 + n = -\chi(\Sigma_{g,n}).$$

Proof. Assume that there exists such a decomposition, and lets determine the numbers h and k.

Let γ_j be one of the curves on $\Sigma_{g,n}$, k_1 the number of connected components of $\Sigma_{g,n} \setminus \gamma_j$, and g_1 the sum of genera of the connected components of $\Sigma_{g,n} \setminus \gamma_j$.

First we observe that

$$g_1 = \begin{cases} g-1 & \text{if } k_1 = 1 \\ g & \text{if } k_1 = 2 \end{cases},$$

and in either case $g_1 - k_1 = g - 2 = (g-1) - 1$. We also notice that $\Sigma_{g,n} \setminus \gamma_j$ has two extra boundary components. If we proceed inductively the genera minus the number of connected components, will be reduced by one for each cutting of a curve. Besides, we gain two extra boundary components. There are h curves, and if we cut along all of them we get $g_h - k_h = (g-1) - h$, $k_h = k$ and $g_h = 0$. In other words $h = k + (g-1)$.

As we noticed, each curve, γ_i, provides the cutting with 2 extra boundary components. Each pair-of-pants has 3 boundary components so $k = \frac{2h+n}{3}$. Therefore $k = 2g - 2 + n = -\chi(\Sigma_{g,n})$ and $h = 3g - 3 + n$.

For the existence, choose g curves, $\{\gamma_i\}$, as in Proposition 6.2. Then $\Sigma_{g,n} \setminus \cup_{i=1}^g \gamma_i$ is a sphere with $2g+n$ boundary components, which is homemorphic to the disk with $2g + n - 1$ holes. If we can create a pair-of-pants decomposition of a disk with m holes, we have a pair-of-pants decomposition of $\Sigma_{g,n}$.

It is not possible to make a pair-of-pants decomposition of a disk or an annulus, because there are not enough boundary components, so m has to be larger than 2. We align the punctures horizontally as in figure 3. If $m = 2$ the surface is itself a pair-of-pants. If $m = 3$ choose a simple closed curve around two of the holes, missing the third. If we cut along this curve, we split our disk in two pairs-of-pants.

If $m \geq 4$ numerate the hole from left to right. Choose a simple closed curve containing only the holes 1 and 2, and a simple closed curve containing only the holes $m-1$ and m. Now for $3 \leq i \leq m-2$ choose concentric simple closed curves containing

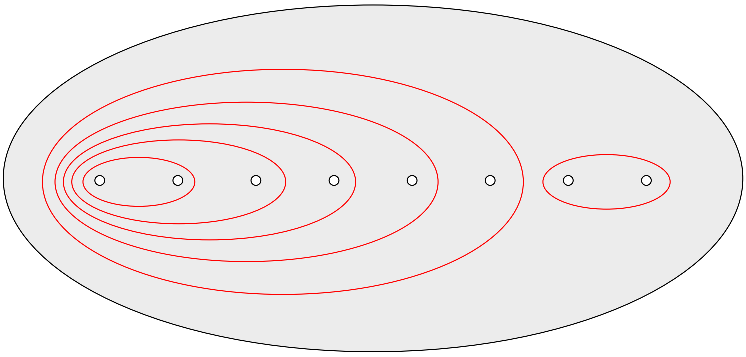


Figure 3: A pair-of-pants decomposition of S^2 with nine holes

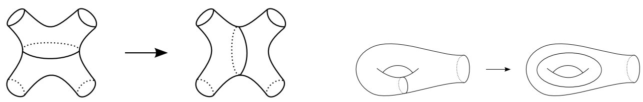
the holes $1, \ldots i$, these curves should be disjoint from other curves, see figure 3. These circles will give a pair-of-pants decomposition of S^2 with $m+1$ boundary components.

The collection of these $3g - 3 + n$ circles give a pair-of-pants decomposition of $\Sigma_{g,n}$.

Through the proof we have noted that surfaces of type $(0, 1)$, $(0, 2)$, and $(1, 0)$ does not have a pair-of-pants decomposition. Jordans Curve Theorem implies that S^2 cannot be decomposed as a collection of type $(0, 3)$ since each choice of a simple closed curve on S^2 will separate it into two disconnected disks. Since we did establish that disks, i.e. surfaces of type $(0, 1)$ cannot be decomposed as a pair-of-pants neither can S^2.

□

Hatcher and Thurston proved, [HT], that any two pair-of-pants decompositions of a surface, $\Sigma_{g,n}$, can be related by a sequence of finitely many moves of type S or A depicted in figure 4.



(a) A-move (b) S-move

Figure 4: An A-move and a S-move

In the beginning of this section, we mentioned that the procedure of cutting objects into pieces, could be used when defining invariants of the objects. With the theorem of Hatcher and Thurston we can build invariants from pair-of-pants decompositions of $\Sigma_{g,n}$. We only have to check invariance under an A- or a S-type move. If it indeed is invariant, it will give the same value for each pair-of-pants decomposition of $\Sigma_{g,n}$ and so, only depend on $\Sigma_{g,n}$.

6.2 Moduli space of a pair-of-pants

This section is a recap of results from [JW, section 3]. Let D be a pair-of-pants, with boundary loops C_1, C_2, C_3, as depicted in figure 5. $\pi_1(D)$ is free of rank 2 since D deformation retracts to a wedge of two circles. We want to assign holonomies around each boundary loop, and hence should use the presentation $\pi_1(D) = \langle [C_1], [C_2], [C_3] \mid [C_1][C_2][C_3] = Id \rangle$.

We define functions $\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3$ on $\mathrm{Hom}(\pi_1(D), SU(2))$ given by

$$\tilde{\theta}_j(\rho) = \cos^{-1}(\frac{1}{2}\operatorname{Tr}\rho([C_j])),$$

which define maps, θ_j, on the moduli space $\mathcal{M}(\Sigma_{g,n})$ into $[0,\pi]$, since Tr is invariant under conjugation.

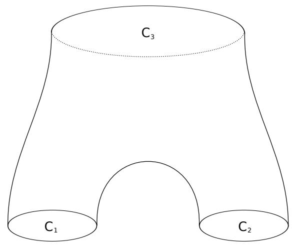


Figure 5: A pair-of-pants

The $SU(2)$ moduli space $\mathcal{M}(\Sigma_{g,n})$ is calculated by showing, that the θ's are injective, and determining their image.

Proposition 6.4. *The map* $\theta = (\theta_1, \theta_2, \theta_3) : \mathcal{M}(\Sigma_{g,n}) \to [0,\pi]^3$ *is a bijection onto the tetrahedron*

$$|\theta_1 - \theta_2| \le \theta_3 \le \min(\theta_1 + \theta_2,\, 2\pi - (\theta_1 + \theta_2))$$

Proof. We represent $SU(2)$ as the unit quaternions, and use the notation $z + wj = \begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix}$, where $z, w \in \mathbb{C}$, $j^2 = -1$ and $zj = j\overline{z}$.

Basically we need to determine, when a triple of angles $(\theta_1, \theta_2, \theta_3) \in [0,\pi]^3$, can be traces of elements in $\mathcal{M}(\Sigma_{g,n})$, and then show that θ is injective.

Let $g_i \in SU(2)$ be an element in the conjugacy class of $e^{i\theta_i} = \begin{bmatrix} e^{i\theta_i} & 0 \\ 0 & e^{-i\theta_i} \end{bmatrix}$, for $i = 1, 2, 3$. Let us pin down the conditions on g_1, g_2, g_3 to satisfy $g_1 g_2 g_3 = 1$.

We can diagonalize g_1, and thus assume that $g_1 = e^{i\theta_1}$, $\theta_1 \in [0,\pi]$. We can conjugate the g_i's even further while keeping g_1 diagonal. This limits our possibilities of conjugation to diagonal matricies $e^{i\varphi}$, $\varphi \in [0,\pi]$. $g_2 = z_2 + w_2 j$, and conjugate it with $e^{i\varphi}$,

$$\tilde{g}_2 = e^{i\varphi}(z_2 + w_2 j)e^{-i\varphi} = z_2 + w_2 e^{2i\varphi} j = z_2 + r_2 e^{i(\psi + 2\varphi)} j.$$

If we choose $\varphi = -\frac{1}{2}\psi$, $\tilde{g}_2 = z_2 + r_2 j$, where $r_2 \in \mathbb{R}_+$. Since $\tilde{g}_2 \in SU(2)$, $|z_2|^2 + r_2^2 = 1$, so $r_2 \in [0,1]$, and we can find $c \in [-1,1]$ satisfying $r_2 = c\sin(\theta_2)$, and since the image of sine is $[-1,1]$, there exists a $\beta \in \mathbb{R}$, such that $c = \sin(\beta)$. Therefore $r_2 = \sin(\beta)\sin(\theta_2)$.

$$1 = |z_2|^2 + r_2^2 = x_2^2 + y_2^2 + \sin(\beta)^2 \sin(\theta_2)^2,$$

where $z_2 = x_2 + iy_2$, $x_2, y_2 \in \mathbb{R}$. This equation is satisfied by $x = \cos(\theta_2)$, $y_2 = \sin(\theta_2)\cos(\beta)$, and thus we can replace g_2 with $\tilde{g}_2 = \cos(\theta_2) + i\sin(\theta_2)\cos(\beta) + j\sin(\theta_2)\sin(\beta) = \cos(\theta_2) + i\sin(\theta_2)(\cos(\beta) - ij\sin(\beta))$.

We have now found a nice expression for both g_1 and g_2, and we could conjugate even more, but that would just ruin the nice expressions, since the only element of $SU(2)$, that leaves g_1, g_2 invariant under simultanious conjugation is ± 1, provided

$\sin\theta_2 \neq 0$. We should now try to determine conditions for when $g_1 g_2$ is conjugate to $e^{i\theta_3}$ – this is the same as solving $g_1 g_2 g_3 = 1$.

Begin with the real part

$$\begin{aligned}\cos\theta_3 =& \operatorname{Re}(e^{i\theta_1}(\cos\theta_2 + i\sin\theta_2(\cos\beta - ij\sin\beta)))\\ =& \operatorname{Re}((\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2(\cos\beta - ij\sin\beta)))\\ =& \operatorname{Re}(\cos\theta_1\cos\theta_2 + i\sin\theta_2\cos\theta_1(\cos\beta - ij\sin\beta) + i\sin\theta_1\cos\theta_2\\ & - \sin\theta_1\sin\theta_2(\cos\beta - ij\sin\beta))\\ =& \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos\beta.\end{aligned}$$

If $\sin\theta_1\sin\theta_2 \neq 0$, this is equivalent to

$$\cos\beta = \frac{\cos\theta_1\cos\theta_2 - \cos\theta_3}{\sin\theta_1\sin\theta_2}. \tag{3}$$

The condition on θ_3, is that the right hand side of equation (3) is in $[-1,1]$. In other words

$$-\sin\theta_1\sin\theta_2 \leq \cos\theta_1\cos\theta_2 - \cos\theta_3 \leq \sin\theta_1\sin\theta_2$$

equivalently

$$\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \leq \cos\theta_3 \leq \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2.$$

which is finally equivalent to

$$\cos(\theta_1+\theta_2) \leq \cos\theta_3 \leq \cos(\theta_1-\theta_2).$$

Since cosine is decreasing on $[0,\pi]$, and if we take into account that cosine is even, the above equation is equvialent to

$$|\theta_1-\theta_2| \leq \theta_3 \leq \min(\theta_1+\theta_2, 2\pi-(\theta_1+\theta_2)), \tag{4}$$

since

$$\begin{aligned}&|\theta_1-\theta_2| \leq \theta_3\\ &\qquad \theta_3 \leq \theta_1+\theta_2 \quad \text{if } |\theta_1+\theta_2| \leq \pi\\ &\qquad \theta_3 \leq 2\pi-(\theta_1+\theta_2) \quad \text{if } |\theta_1+\theta_2| > \pi.\end{aligned}$$

Doing the same kind of calculations for the imaginary part, will give the exact same inequalities.

The inequalities (4) define a tetrahedron in $[0,\pi]^3$ with corners $(0,0,0)$, $(\pi,0,\pi)$, $(\pi,\pi,0)$, and $(0,\pi,\pi)$.

To prove injectivety of θ, observe that $\sin\theta_2\sin\beta \in \mathbb{R}_+$, and if $\sin\theta_1 \neq 0$

$$\sin\theta_2\cos\beta = \frac{\cos\theta_1\cos\theta_2 - \cos\theta_3}{\sin\theta_1},$$

and

$$\tilde{g}_2 = \cos\theta_2 + i\frac{\cos\theta_1\cos\theta_2 - \cos\theta_3}{\sin\theta_1} + j\sin\theta_2\sin\beta.$$

With $\sin\theta_2\sin\beta \geq 0$ and g_1 given, there is a unique solution to the above equation in terms of $\cos\theta_1$, $\cos\theta_2$ and $\cos\theta_3$, and thus also in $\theta_1,\theta_2,\theta_3$ if we restrict the angles to be in $[0,\pi]$. This is also true in the degenerate case $\sin\theta_1 = 0$, because if $\sin\theta_1 = 0$ $\theta_1 = 0$ or $\theta_1 = \pi$ and the choice of a value for either θ_2 or θ_3 determines the other completely. The singular case is exactly the verticies of the tetrahedron. This proves injectivity of θ. The functions θ_i can be considered as coordinates on the space of conjugacy classes of representations of the pair-of-pants fundamental group. □

The general theory of flat connections and representations of the fundamental group of a surface, identifies the flat connections with irreducible representations, and these irreducible representations should be dense in $\mathrm{Hom}(\pi_1(\Sigma), G)/G$.

Corollary 6.5. *Reducible $SU(2)$-represenions of $\pi_1(\Sigma_{0,3})$ is exactly the boundary of the tetrahedron from Proposition 6.4.*

Proof. Assume that $\rho : \pi_1(\Sigma_{0,3}) \to SU(2)$ is a reducible representation, and let's determine its image under θ.

As always, we can choose a representative, (C_1, C_2, C_3), in the conjugacy class of ρ, where $C_1 \in U(1)$. Let $g \in S_\rho$, then

$$g \cdot (C_1, C_2, C_3) = (gC_1g^{-1}, gC_2g^{-1}, gC_3g^{-1}) = (C_1, C_2, C_3).$$

Since $C_1 \in U(1)$ then $g \in U(1)$, so $S_\rho \subset U(1)$. Let $C_2 = z_2 + w_2 j$, then

$$gC_2g^{-1} = e^{i\varphi}(z_2 + w_2 j)e^{-i\varphi} = z_2 + w_2 e^{2i\varphi} j.$$

This is equal to $C_2 = z_2 + w_2 j$ if and only if $w_2 = w_2 e^{2i\varphi}$, which happens exactly when $w_2 = 0$ or $\varphi \in \{0, \pi\}$. If $\varphi \in \{0, \pi\}$ then $g = \pm 1$, and if $w_2 = 0$, $C_2 \in U(1)$. If g is a non-trivial stabilizer, $C_2 \in U(1)$.

In the proof of Proposition 6.4 $r_2 = |w_2|$, and if $|w_2| = 0$, and $C_2 \neq \pm 1$, then $\sin\beta = 0$, and $\cos\beta = \pm 1$, which give equality in one of the inequalities defining the tetrahedron. The irreducible representations are hence mapped to the boundary.

Assume $\theta_3 = \theta_1 - \theta_2$, and show that the corresonding $[(C_1, C_2, C_3)]$ is reducible. We only look at this face since the other three faces of the tetrahedron is treated in the exact same way.

If $\theta_3 = \theta_1 - \theta_2$, $\cos\theta_3 = \cos(\theta_1 - \theta_2) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$, which is equivalent to $-\sin\theta_1\sin\theta_2 = \cos\theta_1\cos\theta_2 - \cos\theta_3$, which again is equivalent to

$$\cos\beta = \frac{\cos\theta_1\cos\theta_2 - \cos\theta_3}{\sin\theta_1\sin\theta_2} = -1.$$

This implies that $\sin\beta = 0$, but $|w_2|^2 = \sin^2\beta\sin^2\theta_2 = 0$, so $C_2 \in U(1)$. Since we can always assume $C_1 \in U(1)$, and C_1, C_2, C_3 to satisfy $C_1C_2C_3 = 1$, and thus $C_3 \in U(1)$. Conclusively $S_{(C_1,C_2,C_3)} \supset U(1)$, and the representation is reducible. □

Remark 6.6. Notice from the proof of Proposition 6.4 that if π_1 has a presentation with three generators A, B, C and some relations, the map $\theta = (\theta_A, \theta_B, \theta_{AB})$, defined by $\theta_A = \cos^{-1}(\frac{1}{2}\mathrm{Tr}(\rho(A)))$, $\theta_B = \cos^{-1}(\frac{1}{2}\mathrm{Tr}(\rho(B)))$ and $\theta_{AB} = \cos^{-1}(\frac{1}{2}\mathrm{Tr}(\rho(AB)))$ are coordiantes on the character variety. In the following we will ignore arccos and only use the matrix trace. The arccos is used in this section to make the moduli space look nicer. The relations between A, B and C can be expressed in these coordinates, and we can then determine the image. This idea will be used to calculate the moduli space of a once punctured torus.

7 Moduli space of the punctured torus

In this section we will calculate the moduli space of $\Sigma_{1,1}$, the once punctured torus. The fundamental group of a torus is $\mathbb{Z} \times \mathbb{Z}$, but with the introduction of a puncture, $\Sigma_{1,1}$, retracts to a wedge of two circles. $\pi_1(\Sigma_{1,1})$ is the free product of rank two, where longitude and meridian of the torus are generators. This presentation is however not convenient for calculating the moduli space. We would like to use the coordiantes from Proposition 6.4, so we introduce an extra generator, and hence also a relation. If we introduce the generator K, which is a curve homotopic to the boundary, we get the relation $KABA^{-1}B^{-1} = Id$. Generally the fundamental group of $\Sigma_{g,n}$, if $g \geq 1$ is generated by $2g+n$ elements, $a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n$, and has a single relation $[b_g, a_g] \dots [b_1, a_1] c_n \dots c_1 = 1$, where $[b_i, a_i] = b_i a_i b_i^{-1} a_i^{-1}$.

As stated in Remark 6.6 the matrix trace providees coordinates on the moduli space. The coordiantes are $x = \operatorname{Tr}(\rho(A))$, $y = \operatorname{Tr}(\rho(B))$, $z = \operatorname{Tr}(\rho(AB))$, where A, B are meridian and longitude on the torus, and ρ is a homomorphism $\pi_1(\Sigma_{1,1}) \to SU(2)$. In this case the moduli space will be a subset of $[-2,2]^3$, and the image will be carved out by the relation from the group presentation.

In stead of specifying the exact image of (x, y, z), it would be more interesting to determine which representations correspond to a specific holonomy along K, i.e. a specific trace of K. We could also have posed this question in the previous example, but it would not give any interesting answers, since specifying a representation on the three boundary components, would in that case determine the representation completely.

Lemma 7.1. *For every $A, B \in SU(2)$ the following trace identity is satisfied*

$$\operatorname{Tr}(ABA^{-1}B^{-1}) = \operatorname{Tr}(A)^2 + \operatorname{Tr}(B)^2 + \operatorname{Tr}(AB)^2 - \operatorname{Tr}(A)\operatorname{Tr}(B)\operatorname{Tr}(AB) - 2.$$

Proof. As always we represent $SU(2)$ as the unit quaternions. Since the matrix trace is invariant under conjugation, we can find an element $g \in SU(2)$ such that $gAg^{-1} = e^{i\theta_A}$ and $gBg^{-1} = re^{i\theta_B} + wj$. Then $gABg^{-1} = re^{i(\theta_A+\theta_B)} + e^{i\theta_A} wj$, and

$$gABA^{-1}B^{-1}g^{-1} = r^2 + |w|^2 \cos(2\theta_A) + di + ej + fk,$$

where d, e, f are real numbers. Then

$$\begin{aligned}
\operatorname{Tr}(ABA^{-1}B^{-1}) &= 2r^2 + 2|w|^2 \cos(2\theta_A) = 2r^2 + 2(1-r^2)\cos(2\theta_A) \\
\operatorname{Tr}(A) &= e^{i\theta_A} + e^{-i\theta_A} \\
\operatorname{Tr}(B) &= re^{i\theta_B} + re^{-i\theta_B} \\
\operatorname{Tr}(AB) &= re^{i(\theta_A+\theta_B)} + re^{-i(\theta_A+\theta_B)},
\end{aligned}$$

where the second equality in the first line is $\det(B) = 1$.

Now we insert the above identities, and reduce the expression.

$$\begin{aligned}
\operatorname{Tr}(A)^2 + &\operatorname{Tr}(B)^2 + \operatorname{Tr}(AB)^2 - \operatorname{Tr}(A)\operatorname{Tr}(B)\operatorname{Tr}(AC) - 2 \\
&= e^{2i\theta_A} + e^{-2i\theta_A} + 2 + r^2 e^{2i\theta_B} + r^2 e^{-2i\theta_B} + 2r^2 + r^2 e^{2i(\theta_A+\theta_B)} + r^2 e^{-2i(\theta_A+\theta_B)} + 2r^2 \\
&\quad - r^2 \big(e^{2i(\theta_A+\theta_B)} + e^{2i\theta_A} + e^{2i\theta_B} + 1 + 1 + e^{-2i\theta_B} + e^{-2i\theta_A} + e^{-2i(\theta_A+\theta_B)}\big) - 2 \\
&= 2\cos(2\theta_A) + 2r^2 - 2r^2\cos(2\theta_A) \\
&= \operatorname{Tr}(ABA^{-1}B^{-1}).
\end{aligned}$$

□

Applying Lemma 7.1 to our case, we have

$$k = \operatorname{Tr}(\rho(K)) = x^2 + y^2 + z^2 - xyz - 2,$$

and if we let

$$\mathcal{M}_k(\Sigma_{1,1}) = \{(x, y, z) \in [-2,2]^3 \mid x^2 + y^2 + z^2 - xyz - 2 = k\},$$

then

$$\mathcal{M}(\Sigma_{1,1}) = \cup_{k\in[-2,2]}\mathcal{M}_k(\Sigma_{1,1}).$$

For $k \in (-2, 2)$ each $\mathcal{M}_k(\Sigma_{1,1})$ is a smooth two-sphere, and the set $M_2(\Sigma_{1,1})$ is a pillow shaped sphere, with 4 singular points in the corners of the pillow, and finally $M_{-2}(\Sigma_{1,1}) = \{(0, 0, 0)\}$. It is very instructive to plot the eqations in say *Mathematica*, and use the *Manipulate* command to see how the shape of the sphere depend on k.

8 The four-punctured sphere

In this section we will the pair-of-pants, to calculate the moduli space of a four-punctured sphere. This section is based on [Gol3] and [Gol2].

The four-punctured sphere can be decomposed as a union of two pair-of-pants, in four different ways, but we only need to look at one decomposition. Let the four boundary components be named A, B, C, D, and the curve $X = AB$. Then the part with boundary (A, B, X), and (C, D, X^{-1}) are the pants.

There are two different ways to proceed from this point. One way illustrates a general principle of glueing together disconnected surfaces, and the other gives a clear picture of this particular moduli space. We will discuss both aspects, and begin with the general glueing procedure.

When a curve ξ in a compact surface, Σ, cuts Σ into two disconnected surfaces, Σ_1, Σ_2, the fundamental group of Σ can be calculated as an amalgamated free product of the fundamental groups $\pi_1(\Sigma_1)$, and $\pi_1(\Sigma_2)$,

$$\pi_1(\Sigma) = \pi_1(\Sigma_1) \sqcup_{\pi_1(\xi)} \pi_1(\Sigma_2).$$

Let $F_i : \pi_1(\xi) \to \pi_1(\Sigma_i) \to \pi_1(\Sigma)$ be induced by inclusion of curves, then the amalgamated free product of $\pi_1(\Sigma_1)$ and $\pi_1(\Sigma_2)$ is the ordinary free product, where we force $F_1(\pi_1(\xi))$ and $F_2(\pi_1(\xi))$ to be equal. In other words

$$\pi_1(\Sigma_1) \sqcup_{\pi_1(\xi)} \pi_1(\Sigma_2) = \pi_1(\Sigma_1) * \pi_1(\Sigma_2)/N,$$

where N is the smallest normal subgroup of the free product, containing all elements of the form $F_1(f)F_2(f)^{-1}$, where $f \in \pi_1(\xi)$.

Let's furthermore require that the homomorphisms F_i, are injective, then the induced maps on homomorphisms are also injective, which will make it easier to determine the moduli space.

In our case $\xi = X$, and we map a generator of $\mathbb{Z} = \pi_1(\xi)$ to $X \in \pi_1(\Sigma_1)$ and $X^{-1} \in \pi_1(\Sigma_1)$.

A representation $\rho : \pi_1(\Sigma_{0,4}) \to SU(2)$ restricts to two representations $\rho_i : \pi_1(\Sigma_i) \to SU(2)$ satisfying $\rho_1(X)\rho_2(X^{-1}) = 1$. Conversely, two representations ρ_1, ρ_2 statisfying $\rho_1(X)\rho_2(X^{-1}) = 1$ define a unique representation ρ of $\pi_1(\Sigma_{0,4})$. This gives an injection

$$\mathrm{Hom}(\pi_1(\Sigma_{0,4}), SU(2)) \hookrightarrow \mathrm{Hom}(\pi_1(\Sigma_1), SU(2)) \times \mathrm{Hom}(\pi_1(\Sigma_2), SU(2)),$$

which induces a map, R, between moduli spaces

$$R : \mathcal{M}(\pi_1(\Sigma_{0,4})) \to \mathcal{M}(\pi_1(\Sigma_1)) \times \mathcal{M}(\pi_1(\Sigma_2)),$$

where the image of R are the elements $([\rho_1], [\rho_2])$ satisfying $[\rho_1]|_{\pi_1(X)} = [\rho_2]|_{\pi_1(X)}$. If we introduce the coordinates from the two pair-of-pants (a, b, x), (c, d, y), e.g. $a = Tr(\rho(A))$, the image of R is $(a, b, x, c, d, y) \in [-2, 2]^6$, where (a, b, x), (c, d, y) satisfy the pair-of-pants inequalities, and furthermore $x = y$.

The map R will not be injective, so to determine the moduli space, we should know the fiber of R at each point.

Assume $(\chi_1, \chi_2) \in \mathrm{Im}(R)$. Let $\rho : \pi_1(\Sigma_{0,4}) \to SU(2)$ be a representation, whose restriction to $\pi_1(\Sigma_i)$, ρ_i, corresponds to χ_i. The centralizer of $\rho(\pi_1(X))$ acts by conjugation on ρ_1, and this action can be used to define a new representation of $\pi_1(\Sigma_{0,4})$. Let $\psi \in \mathcal{Z}(\rho(\pi_1(X)))$ and define

$$T_\psi \rho : \gamma \mapsto \begin{cases} \psi\rho_1(\gamma)\psi^{-1} & \gamma \in \pi_1(\Sigma_1) \\ \rho_2(\gamma) & \gamma \in \pi_1(\Sigma_2) \end{cases}$$

which is a representation of $\pi_1(\Sigma_{0,4})$. It is well-defined since $\psi \in \mathcal{Z}(\rho(\pi_1(X)))$. Furthermore $T_\psi\rho|_{\Sigma_i} = \chi_i$, so $T_\psi\rho \in R^{-1}(\chi_1, \chi_2)$. It can be shown that this action is transitive on each fiber, which describes the content of the fiber. Had the action been free, the fiber would be $\mathcal{Z}(\rho(\pi_1(X)))$.[1]

This construction can be applied any time two disconnected surfaces are glued together along a curve. It is not always clear what the fiber would be, and in this particular case, we have a more direct way of calculating the fiber of R.

Instead of presenting $\pi_1(\Sigma_{0,4})$ as an amalgamation of two groups, we present it by the four generators A, B, C, D and the single relation $ABCD = 1$. We define seven functions on our moduli space, $a = \mathrm{Tr}(\rho(A))$, $b = \mathrm{Tr}(\rho(B))$, $c = \mathrm{Tr}(\rho(C))$, $d = \mathrm{Tr}(\rho(D))$, $x = \mathrm{Tr}(\rho(AB))$, $y = \mathrm{Tr}(\rho(BC))$ and $z = \mathrm{Tr}(\rho(CA))$, ρ is a $SU(2)$-representation of $\pi_1(\Sigma_{0,4})$.

It can be shown, [Mag], that these functions satisfy the equation

$$x^2+y^2+z^2+xyz = (ab+cd)x+(ad+bc)y+(ac+bd)z-(a^2+b^2+c^2+d^2+abcd-4).$$

If we specify the holonomy around A, B, C, D in such a way, that we still have $\rho(ABCD) = 1$, the relative character variety will be those pairs $(x, y, z) \in [-2, 2]^3$, which satisfy the equation for the given $(a, b, c, d) \in [-2, 2]^4$.

If we assume $x \neq \pm 2$, i.e. $\rho(AB) \neq \pm 1$, we can rewrite the equation as

$$\begin{aligned} \frac{2+x}{4}\left((y+z) - \frac{(a+b)(d+c)}{2+x}\right)^2 + \frac{2-x}{4}\left((y-z) - \frac{(a-b)(d-c)}{2-x}\right)^2 \\ = \frac{(x^2-abx+a^2+b^2-4)(x^2-cdx+c^2+d^2-4)}{4-x^2} \end{aligned} \tag{5}$$

Since (a, b, x) and (c, d, x) are characters of a $SU(2)$-representation of $\pi_1(\Sigma_{0,3})$

$$\begin{aligned} 2 &\geq \mathrm{Tr}(ABA^{-1}B^{-1}) = \mathrm{Tr}(AB)^2 + \mathrm{Tr}(A)^2 + \mathrm{Tr}(B)^2 - \mathrm{Tr}(A)\,\mathrm{Tr}(B)\,\mathrm{Tr}(AB) - 2 \\ &= x^2 + a^2 + b^2 - abx - 2 \\ &\geq -2 \end{aligned}$$

In other words

$$\begin{aligned} -4 \leq x^2 + a^2 + b^2 - abx - 4 \leq 0 \\ -4 \leq x^2 + c^2 + d^2 - cdx - 4 \leq 0, \end{aligned}$$

and if (a, b, x) and (c, d, x) correspond to irreducible representations (the non-boundary points of the tetrahedron constituting the moduli space of a pair-of-pants), the right hand side of (5) is positive.

The left hand side of (5) is a quadratic function of y and z. We want to rewrite the left hand side to get a quadratic equation, which describes an ellipse.

[1] I think this is the case, if ρ_1 and ρ_2 are irreducible.

It is an easy calculation to show that equation (5) can be written as

$$Q_x(y - y_0(x), z - z_0(x)) = \frac{(x^2 - abx + a^2 + b^2 - 4)(x^2 - cdx + c^2 + d^2 - 4)}{4 - x^2},$$

where

$$Q_x(\eta, \psi) = \frac{2+x}{4}(\eta + \psi)^2 + \frac{2-x}{4}(\eta - \psi)^2 = \eta^2 + \psi^2 + x\eta\psi, \tag{6}$$

and

$$\begin{aligned} y_0(x) &= \frac{1}{4 - x^2}(2(ad + bc) - x(ac + bd)) \\ z_0(x) &= \frac{1}{4 - x^2}(2(ac + bd) - x(ad + bc)). \end{aligned}$$

Since $x^2 - 4 < 0$ equation (6) is an ellipse. For each (a, b, c, d, x) the fiber of R is an ellipse.

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